Triangles of Absolute Differences

Brian Chen, student, Elementary Division, Bilingual Department,
National Experimental High School of Taiwan,

YunHao Fu, Guangzhou University,

Andy Liu, University of Alberta,

George Sicherman, Wayside, New Jersey,

Herbert Taylor, South Pasadena, California and

Po-Sheng Wu, student, Taipei Municipal Beitou Junior High School.

1. Introduction.

In 1976, George Sicherman thought of the following problem while watching a pool game in Buffalo. Could the fifteen pool balls be arranged in the usual triangular array so that apart from the back row of five numbers, each number in a subsequent row is the absolute difference of the two numbers immediately behind it?

As an illustration, possible arrangements with the pool balls numbered from 1 to 6 and from 1 to 10 are shown below.

1 6 4	4 1 6	6 1 10 8	6 10 1 8
5 2	3 5	5 9 2	4 9 7
3	2	4 7	5 2
		3	3

2 6 5	5 2 6	8 3 10 9	8 10 3 9
4 1	3 4	5 7 1	2 7 6
3	1	2 6	5 1
		4	4

George found a solution to his problem, and established its uniqueness up to reflection about the triangle's vertical axis of symmetry. He communicated the problem to Martin Gardner [2] who published the problem in his famed column *Mathematical Games* in the magazine *Scientific American* in April 1977 [1].

The main result of this paper is that no such arrangements exist which contain more than 15 pool balls.

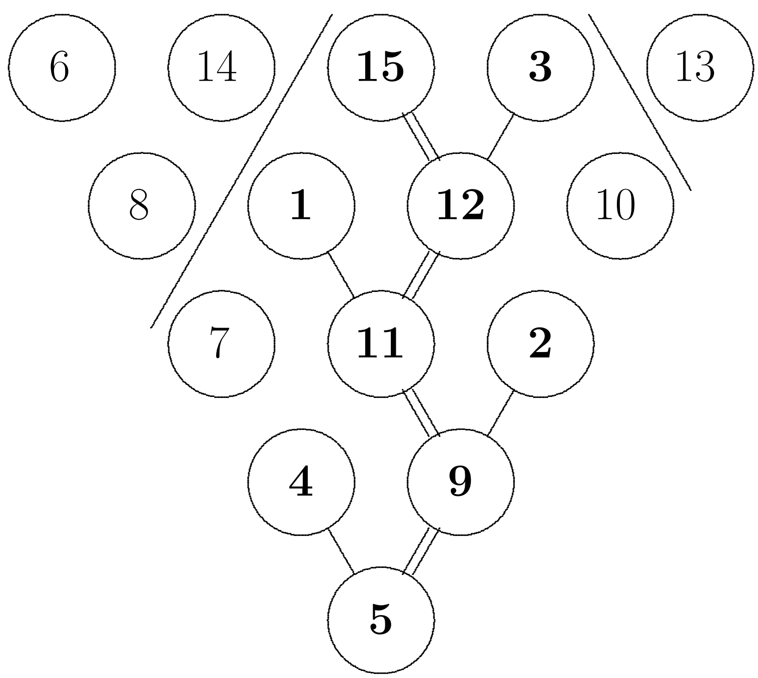
2. The Anatomy of a TOAD.

A **triangle of absolute difference**, or *TOAD* for short, is defined to be a triangular array of integers having the following properties.

1. There are n integers on the top row for some positive integer n.
2. Each row below has one less number than the row above it.
3. Each number below the top row is the absolute difference of the two adjacent numbers in the row immediately above.
4. The integers are from 1 to $\frac{n(n+1)}{2}$, each appearing exactly once.

Such a TOAD is said to be of order n. The reflection of a TOAD about its vertical axis of symmetry is another TOAD. We will treat each pair of mirror twins as a single TOAD. Without loss of generality, we may use either form of the TOAD at our convenience.

Each TOAD has a *spine* defined by the following construction. We call the bottom number of a TOAD its *foot*. From the foot, we draw two line segments connecting it to the two numbers of which it is the absolute difference, a thin line segment to the smaller one and a thick line segment to the larger one. The smaller number is called a *hand* and no further line segments are drawn from it. From the larger number, we draw two more line segments as before. This is continued until the spine reaches the top row. The larger number on the top row is called its *head*. This is illustrated in the diagram below for the unique TOAD of order 5. The spine consists of the numbers 5 (foot), 9, 11, 12 and 15 (head) while the hands consist of the numbers 4, 2, 1 and 3.



A TOAD of order n has a spine consisting of n numbers from foot to head and $n-1$ hands. The head is equal to the sum of the foot and all the hands. Since the head is at most $\frac{n(n+1)}{2}$ while the sum is at least $1+2+\cdots+n=\frac{n(n+1)}{2}$, the head must be equal to $\frac{n(n+1)}{2}$ and the foot and all the hands must collectively be 1, 2, ..., n.

Note that there is exactly one hand or foot on each row, so that the hand or foot is the smallest number of the row. The head is clearly the largest number of the top row. It follows that the largest number on each row is on the spine.

From the head and the top hand, if we draw two slanting lines towards the sides of the TOAD, as shown in the diagram above, we can divide the TOAD into three parts. The two triangles on the side are called quasi-TOADs, with their own feet, spines, heads and hands. They are TOADs except for the fact that they do not have property 4 of the definition of a TOAD.

In the diagram above, the quasi-TOAD on the right is of order 1. Its foot and head coincide (number 13), and it has no hands. The quasi-TOAD on the left is of order 2. The spine connects the foot (number 8) directly to the head (number 14), and it has a single hand (number 6).

For $n \geq 3$, the quasi-TOADs of a TOAD of order n have spines of combined length $n-2$, and a total of $n-2$ feet and hands. This is still true even if one of the quasi-TOADs is empty, which happens if the spine runs along one side of the TOAD.

3. The Nonexistence of TOADs of order 9 or higher.

Suppose we have a TOAD of order $n \geq 3$. The minimum values of the $n-2$ feet and hands of the quasi-TOADs are $n+1,\ n+2,\ \ldots,\ 2n-2$. Hence the sum of their heads is at least $(n+1)+(n+2)+\cdots+(2n-2) = \frac{(n-2)(3n-1)}{2} = \frac{3n^2-7n+2}{2}$.

If there is only one quasi-TOAD, its head is at most $\frac{n(n+1)}{2} - 1$. From $\frac{3n^2-7n+2}{2} \leq \frac{n^2+n-2}{2}$, we have $n^2 \leq 4n-2$. This holds if and only if $n \leq 3$.

If there are two quasi-TOADs, the sum of their heads is at most $\frac{n(n+1)}{2} - 1 + \frac{n(n+1)}{2} - 2$. From $\frac{3n^2-7n+2}{2} \leq n^2+n-3$, we have $n^2 \leq 9n-8$. This holds if and only if $n \leq 8$. It follows that TOADs of order 9 or higher do not exist.

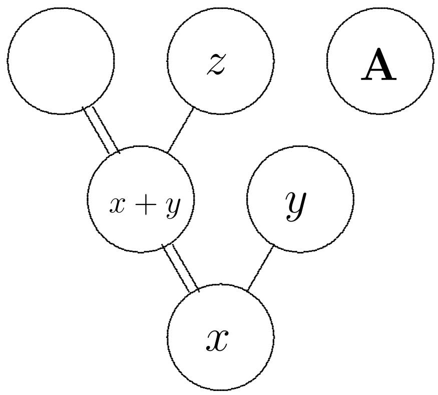
This proof was found in 1977 by Herbert Taylor who communicated it to Martin Gardner [3]. It has not been published until now.

4. The Nonexistence of TOADs of order 8.

For $n = 8$, the equality case holds in the inequality $n^2 \leq 9n - 8$ from the preceding section. If a TOAD of order 8 exists, its head is 36 and its foot and hands are 1, 2, ..., 8. It follows that we have two quasi-TOADs. Their heads are 35 and 34, and their feet and hands are 9, 10, ..., 14. Thus each quasi-TOAD must be of order 3.

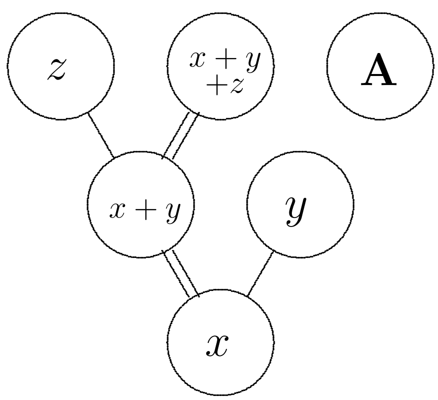
Consider the bottom three rows of the TOAD. Let the foot be x, the hand on the second row from the bottom y and the hand on the third row from the bottom z. Without loss of generality, we may assume that the spine starts off towards the left. There are two cases, according to where the spine intersects the third row from the bottom.

Case 1. The spine continues towards the left.



Since the numbers 1 to 14 form the hands and feet of the TOAD and the two quasi-TOADs, the number $x+y$ must be at least 15. Hence one of x and y is 8 and the other is 7, so that $z \leq 6$. Since $z < y$, we must have A=$y + z \leq 14$. This is a contradiction since A is neither a foot nor a hand of either the TOAD or one of the quasi-TOADs. Thus this case does not yield a TOAD of order 8.

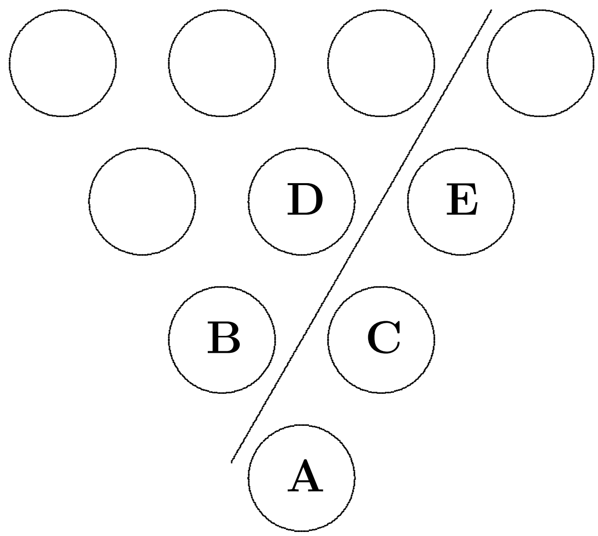
Case 2. The spine turns towards the right.



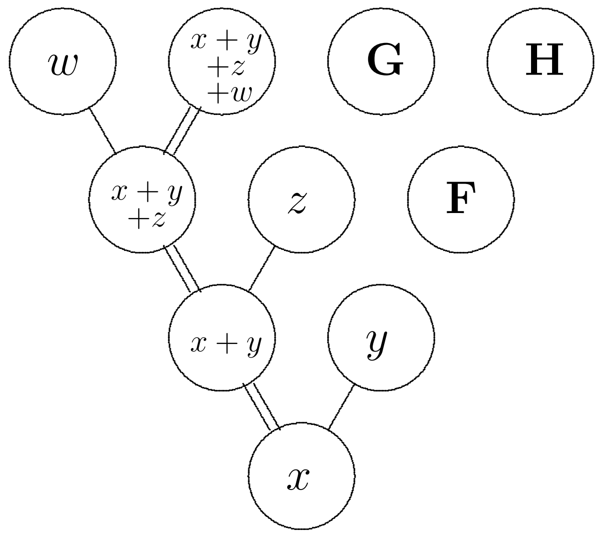
As in Case 1, one of x and y is 8 and the other is 7, so that $z \leq 6$. Since the number on the spine is the largest of the row, we must have A=$(x + y + z) - y = x + z \leq 14$. We have the same contradiction as before. Thus this case does not yield a TOAD of order 8 either.

5. The Nonexistence of TOADs of order 7.

When $n = 7$, the inequality $n^2 \leq 9n - 8$ is no longer tight. Nevertheless, if a TOAD of order 7 exists, we can still make deductions about its structure. Its head is 28 and its foot and hands are 1, 2, ..., 7. Now 8+9+10=27. It follows that one of the quasi-TOADs is of order 3, with 27 as its head and 8, 9 and 10 as its foot and hands, and the remaining two numbers chosen from 17, 18 and 19. Without loss of generality, we may assume that this quasi-TOAD is on the left side, as shown in the diagram below. We consider two cases.



Case 1. C$\geq$ 19.
Then C$\geq$D$\geq$8. Hence E=27 or 28. However, these numbers had been reserved for the heads of the TOAD and the quasi-TOAD of order 3. We have a contradiction.



Case 2. C≤18.
Since B=8, 9 or 10, A must be a hand of the TOAD. It follows that the spine of the TOAD must start off as shown in the diagram above, where x is the foot of the TOAD. Since $z-y<7$, F=$y+z$. Since the head is the largest number on the top row, G=$x+y+w$ and H=$x+w-z$. Each of $y+z$, $x+w-z$ and $x+y$ is at most 13. Since 8, 9 and 10 are the foot and hands of the quasi-TOAD of order 3, these three numbers must be 11, 12 and 13 in some order. However, this means that both the foot and the hand of the quasi-TOAD of order 2 are at least 14, but its head is at most 26. We have a contradiction.

6. The Nonexistence of TOADs of order 6.

In this final section, we abandon the anatomy of a TOAD and turn to other approaches. We present two different parity arguments. Each may be generalized to cover certain cases for $n \geq 9$, but they overlap our main result in Section 3.

The first proof was found in 1976 by George [2]. We work with arithmetic modulo 2 so we may replace differences by sums. The first six rows of the reduced Pascal's Triangle are shown in the diagram below.

```
          1
        1   1
      1   0   1
    1   1   1   1
  1   0   0   0   1
1   1   0   0   1   1
```

Suppose an order 6 TOAD exists. Let the numbers on the top row be $a,\ b,\ c,\ d,\ e$ and f in that order. Then the numbers in the second row are $a+b,\ b+c,\ c+d,\ d+e$ and $e+f$. In modulo 2, those in the third row are $a+c,\ b+d,\ c+e$ and $d+f$, those in the fourth are $a+b+c+d,\ b+c+d+e$ and $c+d+e+f$, those in the fifth $a+e$ and $b+f$, and the only number in the sixth row is $a+b+e+f$. Hence their sum is $6a+8b+8c+8d+8e+6f$, which is even. Since $1+2+\cdots+21=231$ is odd, we have a contradiction.

The second proof was found by Brian Chen in 2008, when he was in Grade 4. Suppose an order 6 TOAD exists. We partition its twenty-one numbers into seven triples, identified by different letters, as shown in the diagram below. Each triple contains an even number of odd numbers. Hence the TOAD contains an even number of odd numbers. However, there are eleven odd numbers from 1 to 21, and we have a contradiction.

```
A   A   B   B   C   C
  A   G   B   G   C
    D   D   E   E
      D   G   E
        F   F
          F
```

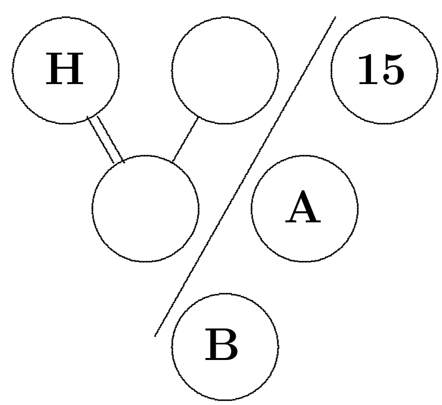
7. TOADs of order 5 or lower.

The unique TOAD of order 1 is trivial, and there are two elementary TOADs of order 2. An order 3 TOAD has an order 1 quasi-TOAD which must be the number 4 or 5. If it is 4, this leads to the TOADs with top rows (4,1,6) and (1,6,4). If it is 5, this leads to the TOADs with top rows (5,2,6) and (2,6,5). Hence there are only four TOADs of order 3.

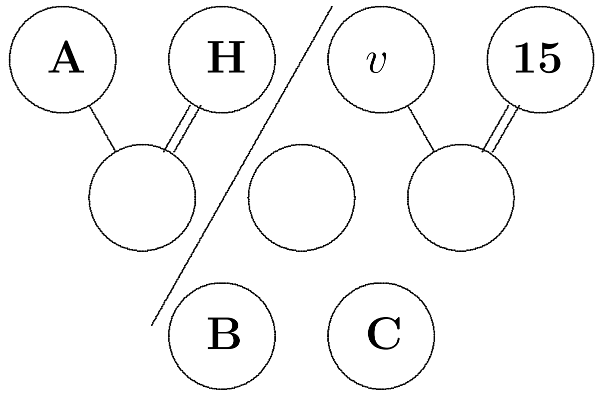
We have proved in Section 3 that any TOAD of order 4 or more must have two non-empty quasi-TOADs. Thus an order 4 TOAD has two quasi-TOADs of order 1. There are four cases.
Case 1. The top hand of the TOAD is 1.
Then 9 is not in the top row, so that 8 must be one of the quasi-TOADs. Whether it appears next to 1 or 10, 7 will not appear in the top row. Hence the other quasi-TOAD must be 6. This leads to the TOADs with top rows (6,1,10,8) and (6,10,1,8) respectively.
Case 2. The top hand is 2.
Then one of the quasi-TOAD must be 9. Whether it appears next to 2 or 10, 8 and 7 will not appear in the top row so that the other quasi-TOAD must be 6. It is routine to verify that we will need two copies of 4 to complete the TOAD.
Case 3. The top hand is 3.
Then the quasi-TOADs must be 8 and 9. This leads to the TOADs with top rows (8,3,10,9) and (8,10,3,9) respectively.
Case 4. The top hand is 4. Then all of 7, 8 and 9 need to be quasi-TOADs, which is clearly impossible.
Hence there are only four TOADs of order 4.

An order 5 TOAD has a quasi-TOAD of order 1 and a quasi-TOAD of order 2. The latter comes from either 8+6=14 or 7+6=13. We may assume that it is on the left side, and let H be its head. There are four cases, according to the positions of H and 15. Note that 15−H=1 or 2.

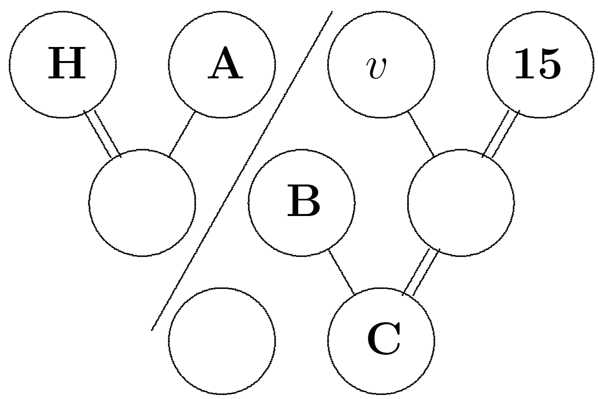
Case 1. H and 15 are in the first and third positions from the left.
A must be 7, 8 or 9, and cannot be a hand. It follows that B cannot be a hand either. However, B must be 1 or 2. This is a contradiction.



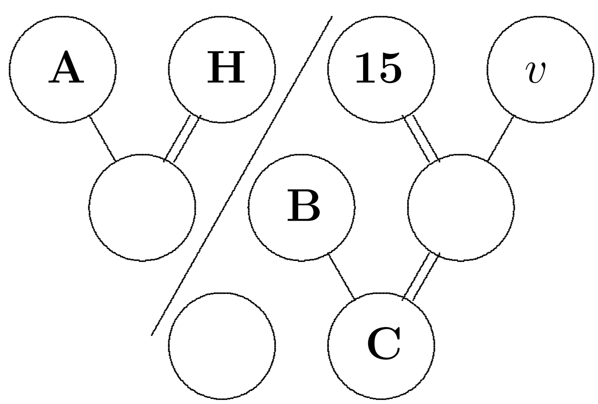
Case 2. H and 15 are in the second and fourth positions from the left.
C is either 1 or 2 and is therefore a hand. In order for B=A−v not to be a hand, we must have A=8 and v=1, but then H=14 and 14 will appear again in the second row from the top. Hence B and C are both hands, a contradiction.



Case 3. H and 15 are in the first and fourth positions from the left.
As in Case 2, B=A−v must be a hand, so that C is on the spine. Now C=15−A must be 9 as otherwise there is no possible value for the number on the spine below it. Now all of the numbers from 10 to 15 must appear in the top two rows, and we are one space short since A, v, H−A and B are all less than 9.



Case 4. H and 15 are in the second and third positions from the left.
We have B=1 or 2, and as in Case 3, C=10, 11 or 12. Routine checking yields the unique solution featured in Section 2.



Bibliography

[1] Martin Gardner, *Penrose Tiles to Trapdoor Ciphers*, second edition, Mathematical Association of America, Washington, (1997) 119–120 and 128–129. See also Martin's *The Colossal Book of Short Puzzles and Problems*, W. W. Norton & Co., New York, (2006) 16–17 and 36–38.

[2] George Sicherman, private communication to Martin Gardner, 1976.

[3] Herbert Taylor, private communication to Martin Gardner, 1977.