Ego Loss May Occur

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Problem Shortlist

Created and Managed by Evan Chen

ELMO regulation:
The problems must be kept strictly confidential until disclosed publicly by the ELMO Committee.

The ELMO 2014 committee gratefully acknowledges the receipt of 43 problems from the following 16 authors:

Ryan Alweiss	3 problems
Matthew Babbitt	1 problem
Evan Chen	3 problems
AJ Dennis	1 problem
Shashwat Kishore	1 problem
Michael Kural	1 problem
Allen Liu	2 problems
Yang Liu	7 problems
Sammy Luo	12 problems
Robin Park	4 problems
Bobby Shen	1 problem
David Stoner	3 problems
Kevin Sun	1 problem
Victor Wang	1 problem
David Yang	1 problem
Jesse Zhang	1 problem

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Part I

Problems

Algebra

A1

A1

In a non-obtuse triangle ABC, prove that

$$\frac{\sin A \sin B}{\sin C} + \frac{\sin B \sin C}{\sin A} + \frac{\sin C \sin A}{\sin B} \geq \frac{5}{2}.$$

Ryan Alweiss

A2

A2

Given positive reals a, b, c, p, q satisfying $abc = 1$ and $p \geq q$, prove that

$$p\left(a^2 + b^2 + c^2\right) + q\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq (p+q)(a+b+c).$$

AJ Dennis

A3

A3

Let a, b, c, d, e, f be positive real numbers. Given that $def + de + ef + fd = 4$, show that

$$((a+b)de + (b+c)ef + (c+a)fd)^2 \geq 12(abde + bcef + cafd).$$

Allen Liu

A4

A4

Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$\begin{aligned} f(x + f(y)) &= g(x) + h(y) \\ g(x + g(y)) &= h(x) + f(y) \\ h(x + h(y)) &= f(x) + g(y) \end{aligned}$$

for all real numbers x and y. (We say a function F is *injective* if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

Evan Chen

A5

A5

Let $\mathbb{R}^*$ denote the set of nonzero reals. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ satisfying

$$f(x^2+y)+1 = f(x^2+1)+\frac{f(xy)}{f(x)}$$

for all $x, y \in \mathbb{R}^*$ with $x^2+y \neq 0$.

Ryan Alweiss

A6

A6

Let a, b, c be positive reals such that $a+b+c = ab+bc+ca$. Prove that

$$(a+b)^{ab-bc}(b+c)^{bc-ca}(c+a)^{ca-ab} \geq a^{ca}b^{ab}c^{bc}.$$

Sammy Luo

A7

A7

Find all positive integers n with $n \geq 2$ such that the polynomial

$$P(a_1, a_2, \ldots, a_n) = a_1^n + a_2^n + \ldots + a_n^n - na_1a_2\ldots a_n$$

in the n variables a_1, a_2, $\ldots$, a_n is irreducible over the real numbers, i.e. it cannot be factored as the product of two nonconstant polynomials with real coefficients.

Yang Liu

A8

A8

Let a, b, c be positive reals with $a^{2014}+b^{2014}+c^{2014}+abc = 4$. Prove that

$$\frac{a^{2013}+b^{2013}-c}{c^{2013}}+\frac{b^{2013}+c^{2013}-a}{a^{2013}}+\frac{c^{2013}+a^{2013}-b}{b^{2013}} \geq a^{2012}+b^{2012}+c^{2012}.$$

David Stoner

A9

A9

Let a, b, c be positive reals. Prove that

$$\sqrt{\frac{a^2(bc+a^2)}{b^2+c^2}}+\sqrt{\frac{b^2(ca+b^2)}{c^2+a^2}}+\sqrt{\frac{c^2(ab+c^2)}{a^2+b^2}} \geq a+b+c.$$

Robin Park

Combinatorics

C1

C1

You have some cyan, magenta, and yellow beads on a non-reorientable circle, and you can perform only the following operations:

1. Move a cyan bead right (clockwise) past a yellow bead, and turn the yellow bead magenta.
2. Move a magenta bead left of a cyan bead, and insert a yellow bead left of where the magenta bead ends up.
3. Do either of the above, switching the roles of the words "magenta" and "left" with those of "yellow" and "right", respectively.
4. Pick any two disjoint consecutive pairs of beads, each either yellow-magenta or magenta-yellow, appearing somewhere in the circle, and swap the orders of each pair.
5. Remove four consecutive beads of one color.

Starting with the circle: "yellow, yellow, magenta, magenta, cyan, cyan, cyan", determine whether or not you can reach a) "yellow, magenta, yellow, magenta, cyan, cyan, cyan", b) "cyan, yellow, cyan, magenta, cyan", c) "magenta, magenta, cyan, cyan, cyan", d) "yellow, cyan, cyan, cyan".

Sammy Luo

C2

C2

A $2^{2014}+1$ by $2^{2014}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct black squares s_1, s_2, $\ldots$, s_n such that s_i and s_{i+1} share an edge for $i = 1, 2, \ldots, n$ (here $s_{n+1} = s_1$).

What is the maximum possible number of filled black squares?

David Yang

C3

C3

We say a finite set S of points in the plane is *very* if for every point X in S, there exists an inversion with center X mapping every point in S other than X to another point in S (possibly the same point).

(a) Fix an integer n. Prove that if $n \geq 2$, then any line segment $\overline{AB}$ contains a unique very set S of size n such that $A, B \in S$.

(b) Find the largest possible size of a very set not contained in any line.

(Here, an *inversion* with center O and radius r sends every point P other than O to the point P' along ray OP such that $OP \cdot OP' = r^2$.)

Sammy Luo

C4

C4

Let r and b be positive integers. The game of *Monis*, a variant of Tetris, consists of a single column of red and blue blocks. If two blocks of the same color ever touch each other, they both vanish immediately. A red block falls onto the top of the column exactly once every r years, while a blue block falls exactly once every b years,

(a) Suppose that r and b are odd, and moreover the cycles are offset in such a way that no two blocks ever fall at exactly the same time. Consider a period of rb years in which the column is initially empty. Determine, in terms of r and b, the number of blocks in the column at the end.

(b) Now suppose r and b are relatively prime and $r+b$ is odd. At time $t=0$, the column is initially empty. Suppose a red block falls at times $t=r, 2r, \dots, (b-1)r$ years, while a blue block falls at times $t=b, 2b, \dots, (r-1)b$ years. Prove that at time $t=rb$, the number of blocks in the column is $|1+2(r-1)(b+r)-8S|$, where

$$S = \left\lfloor \frac{2r}{r+b} \right\rfloor + \left\lfloor \frac{4r}{r+b} \right\rfloor + \dots + \left\lfloor \frac{(r+b-1)r}{r+b} \right\rfloor.$$

Sammy Luo

C5

C5

Let n be a positive integer. For any k, denote by a_k the number of permutations of $\{1, 2, \dots, n\}$ with exactly k disjoint cycles. (For example, if $n=3$ then $a_2=3$ since (1)(23), (2)(31), (3)(12) are the only such permutations.) Evaluate

$$a_n n^n + a_{n-1} n^{n-1} + \dots + a_1 n.$$

Sammy Luo

C6

C6

Let f_0 be the function from $\mathbb{Z}^2$ to $\{0,1\}$ such that $f_0(0,0)=1$ and $f_0(x,y)=0$ otherwise. For each positive integer m, let $f_m(x,y)$ be the remainder when

$$f_{m-1}(x,y) + \sum_{j=-1}^{1} \sum_{k=-1}^{1} f_{m-1}(x+j, y+k)$$

is divided by 2. Finally, for each nonnegative integer n, let a_n denote the number of pairs (x,y) such that $f_n(x,y)=1$. Find a closed form for a_n.

Bobby Shen

Geometry

G1

G1

Let ABC be a triangle with symmedian point K. Select a point A_1 on line BC such that the lines AB, AC, A_1K and BC are the sides of a cyclic quadrilateral. Define B_1 and C_1 similarly. Prove that A_1, B_1, and C_1 are collinear.

Sammy Luo

G2

G2

$ABCD$ is a cyclic quadrilateral inscribed in the circle ω. Let $AB \cap CD = E$, $AD \cap BC = F$. Let ω_1, ω_2 be the circumcircles of AEF, CEF, respectively. Let $\omega \cap \omega_1 = G$, $\omega \cap \omega_2 = H$. Show that AC, BD, GH are concurrent.

Yang Liu

G3

G3

Let $A_1A_2A_3 \cdots A_{2013}$ be a cyclic 2013-gon. Prove that for every point P not the circumcenter of the 2013-gon, there exists a point $Q \neq P$ such that $\frac{A_iP}{A_iQ}$ is constant for $i \in \{1, 2, 3, \cdots, 2013\}$.

Robin Park

G4

G4

Let $ABCD$ be a quadrilateral inscribed in circle ω. Define $E = AA \cap CD$, $F = AA \cap BC$, $G = BE \cap \omega$, $H = BE \cap AD$, $I = DF \cap \omega$, and $J = DF \cap AB$. Prove that GI, HJ, and the B-symmedian are concurrent.

Robin Park

G5

G5

Let P be a point in the interior of an acute triangle ABC, and let Q be its isogonal conjugate. Denote by ω_P and ω_Q the circumcircles of triangles BPC and BQC, respectively. Suppose the circle with diameter $\overline{AP}$ intersects ω_P again at M, and line AM intersects ω_P again at X. Similarly, suppose the circle with diameter $\overline{AQ}$ intersects ω_Q again at N, and line AN intersects ω_Q again at Y.

Prove that lines MN and XY are parallel. (Here, the points P and Q are *isogonal conjugates* with respect to $\triangle ABC$ if the internal angle bisectors of $\angle BAC$, $\angle CBA$, and $\angle ACB$ also bisect the angles $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$, respectively. For example, the orthocenter is the isogonal conjugate of the circumcenter.)

Sammy Luo

G6

G6

Let $ABCD$ be a cyclic quadrilateral with center O. Suppose the circumcircles of triangles AOB and COD meet again at G, while the circumcircles of triangles AOD and BOC meet again at H. Let ω_1 denote the circle passing through G as well as the feet of the perpendiculars from G to AB and CD. Define ω_2 analogously as the circle passing through H and the feet of the perpendiculars from H to BC and DA. Show that the midpoint of GH lies on the radical axis of ω_1 and ω_2.

Yang Liu

G7

G7

Let ABC be a triangle inscribed in circle ω with center O; let ω_A be its A-mixtilinear incircle, ω_B be its B-mixtilinear incircle, ω_C be its C-mixtilinear incircle, and X be the radical center of $\omega_A, \omega_B, \omega_C$. Let A', B', C' be the points at which ω_A, ω_B, ω_C are tangent to ω. Prove that AA', BB', CC' and OX are concurrent.

Robin Park

G8

G8

In triangle ABC with incenter I and circumcenter O, let A', B', C' be the points of tangency of its circumcircle with its A, B, C-mixtilinear circles, respectively. Let ω_A be the circle through A' that is tangent to AI at I, and define ω_B, ω_C similarly. Prove that $\omega_A, \omega_B, \omega_C$ have a common point X other than I, and that $\angle AXO = \angle OXA'$.

Sammy Luo

G9

G9

Let P be a point inside a triangle ABC such that $\angle PAC = \angle PCB$. Let the projections of P onto BC, CA, and AB be X, Y, Z respectively. Let O be the circumcenter of $\triangle XYZ$, H be the foot of the altitude from B to AC, N be the midpoint of AC, and T be the point such that $TYPO$ is a parallelogram. Show that $\triangle THN$ is similar to $\triangle PBC$.

Sammy Luo

G10

G10

We are given triangles ABC and DEF such that $D \in BC, E \in CA, F \in AB$, $AD \perp EF, BE \perp FD, CF \perp DE$. Let the circumcenter of DEF be O, and let the circumcircle of DEF intersect BC, CA, AB again at R, S, T respectively. Prove that the perpendiculars to BC, CA, AB through D, E, F respectively intersect at a point X, and the lines AR, BS, CT intersect at a point Y, such that O, X, Y are collinear.

Sammy Luo

G11

G11

Let ABC be a triangle with circumcenter O. Let P be a point inside ABC, so let the points D, E, F be on BC, AC, AB respectively so that the Miquel point of DEF with respect to ABC is P. Let the reflections of D, E, F over the midpoints of the sides that they lie on be R, S, T. Let the Miquel point of RST with respect to the triangle ABC be Q. Show that $OP = OQ$.

Yang Liu

G12

G12

Let $AB = AC$ in $\triangle ABC$, and let D be a point on segment AB. The tangent at D to the circumcircle ω of BCD hits AC at E. The other tangent from E to ω touches it at F, and $G = BF \cap CD$, $H = AG \cap BC$. Prove that $BH = 2HC$.

David Stoner

G13

G13

Let ABC be a nondegenerate acute triangle with circumcircle ω and let its incircle γ touch AB, AC, BC at X, Y, Z respectively. Let XY hit arcs AB, AC of ω at M, N respectively, and let $P \neq X, Q \neq Y$ be the points on γ such that $MP = MX, NQ = NY$. If I is the center of γ, prove that P, I, Q are collinear if and only if $\angle BAC = 90^\circ$.

David Stoner

Number Theory

N1

N1

Does there exist a strictly increasing infinite sequence of perfect squares $a_1, a_2, a_3, \ldots$ such that for all $k \in \mathbb{Z}^+$ we have that $13^k | a_k + 1$?

Jesse Zhang

N2

N2

Define the Fibanocci sequence recursively by $F_1 = 1$, $F_2 = 1$ and $F_{i+2} = F_i + F_{i+1}$ for all i. Prove that for all integers $b, c > 1$, there exists an integer n such that the sum of the digits of F_n when written in base b is greater than c.

Ryan Alweiss

N3

N3

Let t and n be fixed integers each at least 2. Find the largest positive integer m for which there exists a polynomial P, of degree n and with rational coefficients, such that the following property holds: exactly one of

$$\frac{P(k)}{t^k} \text{ and } \frac{P(k)}{t^{k+1}}$$

is an integer for each $k = 0, 1, \ldots, m$.

Michael Kural

N4

N4

Let $\mathbb{N}$ denote the set of positive integers, and for a function f, let $f^k(n)$ denote the function f applied k times. Call a function $f : \mathbb{N} \to \mathbb{N}$ *saturated* if

$$f^{f^{f(n)}(n)}(n) = n$$

for every positive integer n. Find all positive integers m for which the following holds: every saturated function f satisfies $f^{2014}(m) = m$.

Evan Chen

N5

N5

Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3, 4, 5, 6\}$ and n is a positive integer. Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

Matthew Babbitt

N6

N6

Show that the numerator of

$$\frac{2^{p-1}}{p+1} - \left(\sum_{k=0}^{p-1} \frac{\binom{p-1}{k}}{(1-kp)^2}\right)$$

is a multiple of p^3 for any odd prime p.

Yang Liu

N7

N7

Find all triples (a, b, c) of positive integers such that if n is not divisible by any prime less than 2014, then $n + c$ divides $a^n + b^n + n$.

Evan Chen

N8

N8

Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that:

(i) The greatest common divisor of the sequence $f(1), f(2), \dots$ is 1.

(ii) For all sufficiently large integers n, we have $f(n) \neq 1$ and

$$f(a)^n \mid f(a+b)^{a^{n-1}} - f(b)^{a^{n-1}}$$

for all positive integers a and b.

Yang Liu

N9

N9

Let d be a positive integer and let ε be any positive real. Prove that for all sufficiently large primes p with $\gcd(p-1, d) \neq 1$, there exists an positive integer less than p^r which is not a dth power modulo p, where r is defined by

$$\log r = \varepsilon - \frac{1}{\gcd(d, p-1)}.$$

Shashwat Kishore

N10

N10

Find all positive integer bases $b \geq 9$ so that the number

$$\frac{\overbrace{11\cdots1}^{n-1\ 1's}0\overbrace{77\cdots7}^{n-1\ 7's}8\overbrace{11\cdots1_b}^{n\ 1's}}{3}$$

is a perfect cube in base 10 for all sufficiently large positive integers n.

Yang Liu

N11

N11

Let p be a prime satisfying $p^2 \mid 2^{p-1} - 1$, and let n be a positive integer. Define

$$f(x) = \frac{(x-1)^{p^n} - (x^{p^n} - 1)}{p(x-1)}.$$

Find the largest positive integer N such that there exist polynomials $g(x)$, $h(x)$ with integer coefficients and an integer r satisfying $f(x) = (x-r)^N g(x) + p \cdot h(x)$.

Victor Wang

Part II

Solutions

A2

Given positive reals a, b, c, p, q satisfying $abc = 1$ and $p \geq q$, prove that

$$p\left(a^2+b^2+c^2\right)+q\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq (p+q)(a+b+c).$$

AJ Dennis

Solution 1. First, note it suffices to prove that sum $a^2 + a^{-1}$ is at least twice sum a; in other words, the case $p = q$. Just multiply both sides by q and add $p - q$ times the inequality sum a^2 is at least sum a, which is due to Cauchy and $a + b + c \geq 3$.

So we must show that $a^2+b^2+c^2+1/a+1/b+1/c \geq 2(a+b+c)$. However, we have that $1/a+1/b+1/c \geq 3$ by AM-GM. So it suffices to have $a^2 + b^2 + c^2 + 1 + 1 + 1 \geq 2a + 2b + 2c$, but $a^2 + 1 \geq 2a$ and similar so this is obvious. ■

Solution 2. Note $\sum a^2 \geq \sum bc = \sum a^{-1}$ by AM-GM (or Cauchy-Schwarz), so $LHS \geq \frac{p+q}{2}\left(\sum a^2 + \sum bc\right)$. But

$$\sum a^2 + \sum bc = \sum(a^2 + \frac{1}{2}(ab+ac)) \geq 2\sum a^{3/2}b^{1/4}c^{1/4} = 2\sum a^{5/4}$$

Now we can finish by weighted AM-GM or (weighted) CS/Holder to get $\sum a^{5/4} \geq \sum a$, implying the result. ■

This problem and its solutions were proposed by AJ Dennis.

A3

Let a, b, c, d, e, f be positive real numbers. Given that $def + de + ef + fd = 4$, show that

$$((a+b)de + (b+c)ef + (c+a)fd)^2 \geq 12(abde + bcef + cafd).$$

Allen Liu

Solution 1. First, some beginning stuff. Note that the condition implies that $d = \frac{2m}{n+p}, e = \frac{2n}{m+p}, f = \frac{2p}{m+n}$ $(*)$.

Also, the inequality $(a+b+c)^2 \geq (2\cos(X)+2)\cdot ab + (2\cos(Y)+2)\cdot ac + (2\cos(Z)+2)\cdot bc$, where X, Y, Z are angles of a triangle. (Note hard, just use quadratic discriminants).

Now rewrite the LHS as $(a(de+df) + b(de+ef) + c(df+ef))^2$ and then substitute $A = a(de+df), B = b(de+ef), C = c(df+ef)$. Then, the inequality becomes $(A+B+C)^2 \geq 12\sum_{cyc}\frac{BC}{(d+e)(d+f)}$. So now it suffices to find a triangle such that

$$\frac{12}{(d+e)(d+f)} \leq 2\cos(X) + 2$$

and its cyclic counterparts hold. But note that if the triangle has side lengths $y+z, x+z, x+y$, then $2\cos(X) + 2 = 4\frac{x(x+y+z)}{(x+y)(x+z)}$.

So we need

$$\frac{3}{(d+e)(d+f)} \leq \frac{x(x+y+z)}{(x+y)(x+z)}$$

So substitute in $(*)$ to get the equivalent statement

$$\frac{3(m+n)(m+p)(n+p)^2}{(m^2+mp+n^2+np)(m^2+mn+p^2+np)} \leq 4\frac{x(x+y+z)}{(x+y)(x+z)}$$

So choose $x = np(n+p), y = mp(m+p), z = mn(m+n)$. It is not hard to show that the above inequality reduces to

$$4\left(mn(m+n) + mp(m+p) + np(n+p)\right) \geq 3(m+n)(m+p)(n+p)$$

, which is immediate by expansion. ■

This problem and solution were proposed by Allen Liu.

Solution 2. Note that $de + ef + fe \geq 3$, so we have:

$$(e+f)^2(d+f)(e+d) \geq (3+d^2)(e+f)^2$$

$$\implies [(e+f)(d+f)-3][(e+d)(e+f)-3] \geq [3-d(e+f)]^2$$

Therefore,

$$4\left[\frac{1}{(e+f)(d+f)} - \frac{3}{(e+f)^2(d+f)^2}\right]\left[\frac{1}{(e+d)(e+f)} - \frac{3}{(e+d)^2(e+f)^2}\right]$$
$$\geq \left[\frac{1}{(d+f)(f+e)} + \frac{1}{(d+e)(e+f)} - \frac{1}{(d+e)(d+f)} - \frac{6}{(d+e)(d+f)(e+f)^2}\right]^2$$

Therefore the quadratic expression:

$$y^2\left[\frac{1}{(e+f)(d+f)} - \frac{3}{(e+f)^2(d+f)^2}\right]$$
$$+yz\left[\frac{1}{(d+f)(f+e)} + \frac{1}{(d+e)(e+f)} - \frac{1}{(d+e)(d+f)} - \frac{6}{(d+e)(d+f)(e+f)^2}\right]$$
$$+z^2\left[\frac{1}{(e+d)(e+f)} - \frac{3}{(e+d)^2(e+f)^2}\right]$$

is always nonnegative. (The y^2 and constant coefficients are positive). So:

$$(y+z)\left[\frac{y}{(d+f)(e+f)}+\frac{z}{(d+e)(e+f)}\right]\ge \frac{yz}{e+f}+3\left(\frac{y}{(d+e)}+\frac{z}{(d+f)}\right)^2$$
$$\Longrightarrow 4\left[(y+z)^2-\frac{12yz}{(d+e)(d+f)}\right]\ge\left[2(y+z)-\frac{12y}{(d+f)(e+f)}-\frac{12z}{(d+e)(e+f)}\right]^2.$$

So the quadratic expression:

$$x^2+x\left[2y+2z-\frac{12y}{(d+f)(e+f)}-\frac{12z}{(f+e)(e+f)}\right]+y^2+c^2+2yz-\frac{12yz}{(d+e)(d+f)}$$

is always nonnegative. (The x^2 and constant coefficients are positive). So:

$$(x+y+z)^2\ge\sum_{\text{cyc}}\frac{x}{(d+e)(d+f)}$$

which is precisely what we want to show. (Let $x=a(de+df)$, et cetera.) ■

This second solution was suggested by David Stoner.

A4

Find all triples (f, g, h) of injective functions from the set of real numbers to itself satisfying

$$f(x+f(y)) = g(x)+h(y)$$
$$g(x+g(y)) = h(x)+f(y)$$
$$h(x+h(y)) = f(x)+g(y)$$

for all real numbers x and y. (We say a function F is *injective* if $F(a) \neq F(b)$ for any distinct real numbers a and b.)

Evan Chen

Answer. For all real numbers x, $f(x) = g(x) = h(x) = x + C$, where C is an arbitrary real number.

Solution 1. Let a, b, c denote the values $f(0)$, $g(0)$ and $h(0)$. Notice that by putting $y = 0$, we can get that $f(x + a) = g(x) + c$, etc. In particular, we can write

$$h(y) = f(y - c) + b$$

and

$$g(x) = h(x - b) + a = f(x - b - c) + a + b$$

So the first equation can be rewritten as

$$f(x + f(y)) = f(x - b - c) + f(y - c) + a + 2b.$$

At this point, we may set $x = y - c - f(y)$ and cancel the resulting equal terms to obtain

$$f(y - f(y) - (b + 2c)) = -(a + 2b).$$

Since f is injective, this implies that $y - f(y) - (b + 2c)$ is constant, so that $y - f(y)$ is constant. Thus, f is linear, and $f(y) = y + a$. Similarly, $g(x) = x + b$ and $h(x) = x + c$.

Finally, we just need to notice that upon placing $x = y = 0$ in all the equations, we get $2a = b + c$, $2b = c + a$ and $2c = a + b$, whence $a = b = c$.

So, the family of solutions is $f(x) = g(x) = h(x) = x + C$, where C is an arbitrary real. One can easily verify these solutions are valid. ■

This problem and solution were proposed by Evan Chen.

Remark. Although it may look intimidating, this is not a very hard problem. The basic idea is to view $f(0)$, $g(0)$ and $h(0)$ as constants, and write the first equation entirely in terms of $f(x)$, much like we would attempt to eliminate variables in a standard system of equations. At this point we still had two degrees of freedom, x and y, so it seems likely that the result would be easy to solve. Indeed, we simply select x in such a way that two of the terms cancel, and the rest is working out details.

Solution 2. First note that plugging $x = f(a), y = b; x = f(b), y = a$ into the first gives $g(f(a)) + h(b) = g(f(b)) + h(a) \implies g(f(a)) - h(a) = g(f(b)) - h(b)$. So $g(f(x)) = h(x) + a_1$ for a constant a_1. Similarly, $h(g(x)) = f(x) + a_2, f(h(x)) = g(x) + a_3$.

Now, we will show that $h(h(x)) - f(x)$ and $h(h(x)) - g(x)$ are both constant. For the second, just plug in $x = 0$ to the third equation. For the first, let $x = a_3, y = k$ in the original to get $g(f(h(k))) = h(a_3) + f(k)$. But $g(f(h(k))) = h(h(k)) + a_1$, so $h(h(k)) - f(k) = h(a_3) - a_1$ is constant as desired.

Now $f(x) - g(x)$ is constant, and by symmetry $g(x) - h(x)$ is also constant. Now let $g(x) = f(x) + p, h(x) = f(x) + q$. Then we get:

$$f(x + f(y)) = f(x) + f(y) + p + q$$
$$f(x + f(y) + p) = f(x) + f(y) + q - p$$
$$f(x + f(y) + q) = f(x) + f(y) + p - q$$

Now plugging in (x, y) and (y, x) into the first one gives $f(x+f(y)) = f(y+f(x)) \implies f(x)-x = f(y)-y$ from injectivity, $f(x) = x + c$. Plugging this in gives $2p = q, 2q = p, p + q = 0$ so $p = q = 0$ and $f(x) = x + c, g(x) = x + c, h(x) = x + c$ for a constant c are the only solutions. ■

This second solution was suggested by David Stoner.

Solution 3. By putting $(x, y) = (0, a)$ we derive that $f(f(a)) = g(0) + h(a)$ for each a, and the analogous counterparts for g and h. Thus we can derive from $(x, y) = (t, g(t))$ that

$$\begin{aligned} h(f(t) + h(g(t))) &= f(f(t)) + g(g(t)) \\ &= g(0) + h(t) + h(0) + f(t) \\ &= f(f(0)) + g(t + g(t)) \\ &= h(f(0) + h(t + g(t))) \end{aligned}$$

holds for all t. Thus by injectivity of h we derive that

$$f(x) + h(g(x)) = f(0) + h(x + g(x)) \quad (*)$$

holds for every x.

Now observe that placing $(x, y) = (g(a), a)$ gives

$$g(2g(a)) = g(g(a) + g(a)) = h(g(a)) + f(a)$$

while placing $(x, y) = (g(a) + a, 0)$ gives

$$g(g(a) + a + g(0)) = h(a + g(a)) + f(0).$$

Equating this via $(*)$ and applying injectivity of g again, we find that

$$2g(a) = g(a) + a + g(0)$$

for each a, whence $g(x) = x + b$ for some real number b. We can now proceed as in the earlier solutions. ■

This third solution was suggested by Mehtaab Sawhney.

Solution 4. In the first given, let $x = a + g(0)$ and $y = b$ to obtain

$$f(a + g(0) + f(b)) = g(a + g(0)) + h(b) = h(a) + h(b) + f(0).$$

Swapping the roles of a and b, we discover that

$$f(b + g(0) + f(a)) = f(a + g(0) + f(b)).$$

But f is injective; this implies $f(x) - x$ is constant, and we can the proceed as in the previous solutions. ■

This fourth solution was suggested by alibez.

A6

Let a, b, c be positive reals such that $a+b+c = ab+bc+ca$. Prove that

$$(a+b)^{ab-bc}(b+c)^{bc-ca}(c+a)^{ca-ab} \geq a^{ca}b^{ab}c^{bc}.$$

Sammy Luo

Solution 1. Note $f(x) = x\log x$ is convex. The key step: weighted Popoviciu gives

$$bf(a)+cf(b)+af(c)+(a+b+c)f\left(\frac{bc+ca+ab}{a+b+c}\right) \geq \sum_{\text{cyc}}(b+c)f\left(\frac{ab+bc}{b+c}\right).$$

Exponentiating gives

$$a^{ab}\cdot b^{bc}\cdot c^{ca}\cdot\left(\frac{bc+ca+ab}{a+b+c}\right)^{bc+ca+ab} \geq \prod_{\text{cyc}}\left(\frac{b(c+a)}{b+c}\right)^{bc+ab}$$

$$= \prod_{\text{cyc}} a^{ab+ca}(b+c)^{ab+ca-bc-ab}$$

Cancelling some terms and using $\frac{bc+ca+ab}{a+b+c} = 1$ gives

$$1 \geq \prod_{\text{cyc}} a^{ca}(a+b)^{bc-ab}$$

which rearranges to the result. ■

This problem and solution were proposed by Sammy Luo.

Solution 2. Let $a+b+c = ab+bc+ca = S$. We have

$$\prod_{\text{cyc}}\left(\frac{b(a+c)}{a+b}\right)^{ab} \leq \frac{1}{S}\sum_{\text{cyc}}\frac{ab^2(a+c)}{a+b} \leq 1$$

Where the last is true because:

$$(ab+bc+ca)^2-(a+b+c)\left[\sum_{\text{cyc}}\frac{ab^2(a+c)}{a+b}\right] = \frac{abc(\sum_{\text{cyc}}a^3b-\sum a^2bc)}{(a+b)(b+c)(c+a)} \geq 0$$

as desired. ■

This second solution was suggested by David Stoner.

A7

Find all positive integers n with $n \geq 2$ such that the polynomial

$$P(a_1, a_2, \ldots, a_n) = a_1^n + a_2^n + \ldots + a_n^n - na_1a_2 \ldots a_n$$

in the n variables a_1, a_2, $\ldots$, a_n is irreducible over the real numbers, i.e. it cannot be factored as the product of two nonconstant polynomials with real coefficients.

Yang Liu

Answer. The permissible values are $n = 2$ and $n = 3$.

Solution. For $n = 2$ and $n = 3$ we respectively have the factorizations $(a_1 - a_2)^2$ and

$$\tfrac{1}{2}(a_1 + a_2 + a_3)(a_1^2 + a_2^2 + a_3^2 - a_1a_2 - a_2a_3 - a_3a_1).$$

For $n \geq 4$, we view P at as a polynomial in a_1 and note that the constant term is $a_2^n + a_3^n + \ldots + a_n^n$. So this polynomial must be reducible. We can set $a_5, a_6, \ldots, a_n = 0$, so now we need for $a_2^n + a_3^n + a_4^n$ to be irreducible over $\mathbb{C}$. Let $a = a_2$, $b = a_3$, $c = a_4$. Now we look at it as a polynomial in a, and it factors as

$$\prod_{i=1}^{n} \left(a + \omega_i \cdot \sqrt[n]{b^n + c^n}\right)$$

where the ω_i are the necessary roots of unity. Now we look how we can split this into two polynomials and look at their respective constant terms. So the constant terms would be $\omega(b^n + c^n)^{\frac{k}{n}}$ for some $0 < k < n$, and some root of unity ω. So the previous expression must be a polynomial, say $Q(x)$. But $(b^n + c^n)^k = Q(x)^n$. On the right-hand side, each root has multiplicity n, but since $b^n + c^n$ has no double roots, all roots on the left-hand side have multiplicity $k < n$, contradiction. ■

This problem and solution were proposed by Yang Liu.

A8

Let a, b, c be positive reals with $a^{2014} + b^{2014} + c^{2014} + abc = 4$. Prove that

$$\frac{a^{2013} + b^{2013} - c}{c^{2013}} + \frac{b^{2013} + c^{2013} - a}{a^{2013}} + \frac{c^{2013} + a^{2013} - b}{b^{2013}} \geq a^{2012} + b^{2012} + c^{2012}.$$

David Stoner

Solution. The problem follows readily from the following lemma.

Lemma 1. *Let x, y, z be positive reals, not all strictly on the same side of 1. Then $\sum \frac{x}{y} + \frac{y}{x} \geq \sum x + \frac{1}{x}$.*

Proof. WLOG $(x-1)(y-1) \leq 0$; then

$$(x + y + z - 1)(x^{-1} + y^{-1} + z^{-1} - 1) \geq (xy + z)(x^{-1}y^{-1} + z) \geq 4$$

by Cauchy. Alternatively, if $x, y \geq 1 \geq z$, one may smooth z up to 1 (e.g. by differentiating with respect to z and observing that $x^{-1} + y^{-1} - 1 \leq x + y - 1$) to reduce the inequality to $\frac{x}{y} + \frac{y}{x} \geq 2$. □

Now simply note that $\sum a^{2013} + a^{-2013} \geq \sum a^{2012} + a^{-2012}$. ■

This problem and solution were proposed by David Stoner.

Remark. An earlier (and harder) version of the problem asked to prove that

$$\left(\sum_{\text{cyc}} a(a^2 + bc)\right)\left(\sum_{\text{cyc}}\left(\frac{a}{b} + \frac{b}{a}\right)\right) \geq \left(\sum_{\text{cyc}} \sqrt{(a+1)(a^3 + bc)}\right)\left(\sum_{\text{cyc}} \sqrt{a(a+1)(a+bc)}\right).$$

However, it was vetoed by the benevolent dictator.

Here is the solution to the harder version. Let $s_i = a^i + b^i + c^i$ and $p = abc$. The key is to Cauchy out s_3's from the RHS and use the lemma (in the form $s_1 s_{-1} - 3 \geq s_1 + s_{-1}$) on the LHS to reduce the problem to

$$(s_1 + s_{-1})^2(s_3 + 3p)^2 \geq (3 + s_1)(3 + s_{-1})(s_3 + ps_{-1})(s_3 + ps_1).$$

By AM-GM on the RHS, it suffices to prove

$$\frac{\frac{s_1+s_{-1}}{2} + \frac{s_1+s_{-1}}{2}}{\frac{s_1+s_{-1}}{2} + 3} \geq \frac{s_3 + p\frac{s_1+s_{-1}}{2}}{s_3 + 3p},$$

or equivalently, since $\frac{s_1+s_{-1}}{2} \geq 3$, that $\frac{s_3}{p} \geq \frac{s_1+s_{-1}}{2}$. By the lemma, this boils down to $2\sum_{\text{cyc}} a^3 \geq \sum_{\text{cyc}} a(b^2 + c^2)$, which is obvious.

A9

Let a, b, c be positive reals. Prove that

$$\sqrt{\frac{a^2(bc+a^2)}{b^2+c^2}}+\sqrt{\frac{b^2(ca+b^2)}{c^2+a^2}}+\sqrt{\frac{c^2(ab+c^2)}{a^2+b^2}} \ge a+b+c.$$

Robin Park

Remark. Equality occurs not only at $a=b=c$ but also when $a=b$ and $c=0$.

Solution. By Holder,

$$\left(\sum_{\text{cyc}}\sqrt{\frac{a^2(a^2+bc)}{b^2+c^2}}\right)^2\left(\sum_{\text{cyc}} a(a^2+bc)^2(b^2+c^2)\right) \ge \left(\sum_{\text{cyc}} a(a^2+bc)\right)^3.$$

So we need to prove that

$$\left(\sum_{\text{cyc}} a(a^2+bc)\right)^3 \ge (a+b+c)^2\left(\sum_{\text{cyc}} a(a^2+bc)^2(b^2+c^2)\right).$$

Expanding this gives the following triangle in Chinese Dumbass Notation.

```
                  1
                0   0
              -1  9  -1
            1  -3  -3  1
          -1  -8  21  -8  -1
        -1  8  -11  -11  8  -1
      1  -8  -11  21  -11  -8  1
    -1  -3  21  -11  -11  21  -3  1
   0  9  -3  -8  8  -8  -3  9  0
 1  0  -1  1  -1  -1  1  -1  0  1
```

This is the sum of the following seven inequalities:

$$0 \le \sum_{\text{cyc}} a^5(a^2-b^2)(a^2-c^2)$$

$$0 \le \sum_{\text{cyc}} b^3c^3(b+c)(b-c)^2$$

$$0 \le \sum_{\text{cyc}} 3abc\cdot a^4(a-b)(a-c)$$

$$0 \le \sum_{\text{cyc}} 2abc\cdot a^2(a^2-b^2)(a^2-c^2)$$

$$0 \le \sum_{\text{cyc}} 2abc\cdot (b^4+c^4+2bc(b^2+c^2))(b-c)^2$$

$$0 \le \sum_{\text{cyc}} 17(abc)^2\cdot a(a-b)(a-c)$$

$$0 \le \sum_{\text{cyc}} 6(abc)^2\cdot a(b-c)^2.$$

Hence we're done. ■

This problem was proposed by Robin Park. This solution was given by Evan Chen.

C1

You have some cyan, magenta, and yellow beads on a non-reorientable circle, and you can perform only the following operations:

1. Move a cyan bead right (clockwise) past a yellow bead, and turn the yellow bead magenta.
2. Move a magenta bead left of a cyan bead, and insert a yellow bead left of where the magenta bead ends up.
3. Do either of the above, switching the roles of the words "magenta" and "left" with those of "yellow" and "right", respectively.
4. Pick any two disjoint consecutive pairs of beads, each either yellow-magenta or magenta-yellow, appearing somewhere in the circle, and swap the orders of each pair.
5. Remove four consecutive beads of one color.

Starting with the circle: "yellow, yellow, magenta, magenta, cyan, cyan, cyan", determine whether or not you can reach a) "yellow, magenta, yellow, magenta, cyan, cyan, cyan", b) "cyan, yellow, cyan, magenta, cyan", c) "magenta, magenta, cyan, cyan, cyan", d) "yellow, cyan, cyan, cyan".

Sammy Luo

Solution. So represent the beads in a string; write j for ma[u]j[/u]enta, i for [u]i[/u]ellow, C for cyan. Also, write k as a shorthand for ij, and 1 for (no beads). So $Ci = jC, Cj = kC, Ck = iC$. Also, $iiii = jjjj = 1$, $ij...ij = ji...ji$

We are reminded of quaternion multiplication. So what's C? We could ignore this question by moving all the Cs together; instead, we interpret the string as a series of operations (applied from left to right) to perform on a quaternion. Note that if a yellow bead corresponds to left multiplying by i and a magenta bead by j, i.e. an i in the string transforms $x = a + bi + cj + dk$ to $ix = -b + ai - dj + ck$, where $a, b, c, d \in \mathbb{R}$, then the operation $C(x) = a + ci + dj + bk$ that cyclicly permutes the i, j, k components satisfies

$$i(C(x)) = -c + ai - bj + dk = C(-c + di + aj - bk) = C(j(x)).$$

So $Ci = jC$ in the beads; similarly, $Cj = kC, Ck = iC$ as wanted.

So we let this be the cyan operation. Then, starting with the general quaternion $x = a + bi + cj + dk$, the initial state of the bead string, $iijjCCC$, gives $C(C(C(j(j(i(i(x))))))) = x$, since $C^3 = 1$. Since all the beads are invertible, starting the string at any other place in the circle will still produce the identity; all the allowed bead operations preserve the fact that the bead string composes to an identity (since removing 4 cyan beads will never be possible). Now we can check that the other strings do not compose to the identity.

- The first one is $ijijCCC$ which is multiplication by -1.
- The second is $CiCjC = jCkCC = jiCCC$, which is left multiplication by k.
- The third is $jjCCC$, again multiplication by -1.
- The fourth is $iCCC$, left multiplication by i.

So all are impossible. ■

This problem and solution were proposed by Sammy Luo.

C2

A $2^{2014}+1$ by $2^{2014}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct black squares $s_1, s_2, \ldots, s_n$ such that s_i and s_{i+1} share an edge for $i = 1, 2, \ldots, n$ (here $s_{n+1} = s_1$).

What is the maximum possible number of filled black squares?

David Yang

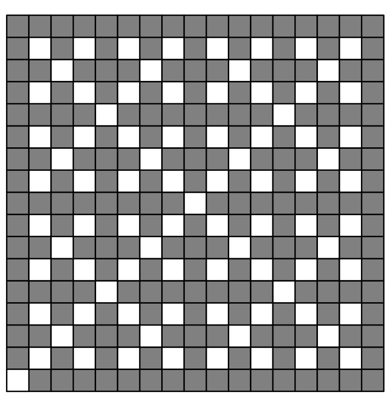
Answer. If $n = 2^m + 1$ is the dimension of the grid, the answer is $\frac{2}{3}n(n+1) - 1$. In this particular instance, $m = 2014$ and $n = 2^{2014} + 1$.

Solution 1. Let $n = 2^m + 1$. Double-counting square edges yields $3v + 1 \le 4v - e \le 2n(n+1)$, so because $n \not\equiv 1 \pmod 3$, $v \le 2n(n+1)/3 - 1$. Observe that if $3 \nmid n - 1$, equality is achieved iff (a) the graph formed by black squares is a connected forest (i.e. a tree) and (b) all but two square edges belong to at least one black square.

We prove by induction on $m \ge 1$ that equality can in fact be achieved. For $m = 1$, take an "H-shape" (so if we set the center at $(0,0)$ in the coordinate plane, everything but $(0, \pm 1)$ is black); call this G_1. To go from G_m to G_{m+1}, fill in $(2x, 2y)$ in G_{m+1} iff (x, y) is filled in G_m, and fill in (x, y) with x, y not both even iff $x + y$ is odd (so iff one of x, y is odd and the other is even). Each "newly-created" white square has both coordinates odd, and thus borders 4 (newly-created) black squares. In particular, there are no new white squares on the border (we only have the original two from G_1). Furthermore, no two white squares share an edge in G_{m+1}, since no square with odd coordinate sum is white. Thus G_{m+1} satisfies (b). To check that (a) holds, first we show that $(2x_1, 2y_1)$ and $(2x_2, 2y_2)$ are connected in G_{m+1} iff (x_1, y_1) and (x_2, y_2) are black squares (and thus connected) in G_m (the new black squares are essentially just "bridges"). Indeed, every path in G_{m+1} alternates between coordinates with odd and even sum, or equivalently, new and old black squares. But two black squares (x_1, y_1) and (x_2, y_2) are adjacent in G_m iff $(x_1 + x_2, y_1 + y_2)$ is black and adjacent to $(2x_1, 2y_1)$ and $(2x_2, 2y_2)$ in G_{m+1}, whence the claim readily follows. The rest is clear: the set of old black squares must remain connected in G_{m+1}, and all new black squares (including those on the boundary) border at least one (old) black square (or else G_m would not satisfy (b)), so G_{m+1} is fully connected. On the other hand, G_{m+1} cannot have any cycles, or else we would get a cycle in G_m by removing the new black squares from a cycle in G_{m+1} (as every other square in a cycle would have to have odd coordinate sum). ■

This problem and solution were proposed by David Yang.

Solution 2. As above, we can show that there are at most $\frac{2}{3}n(n+1) - 1$ black squares. We provide a different construction now for $n = 2^k + 1$.

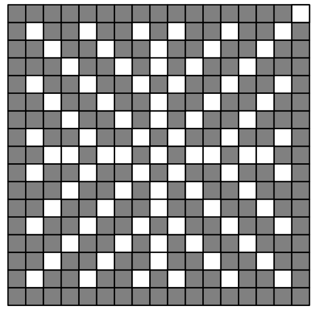


Consider the grid as a coordinate plane (x, y) where $0 \le x, y \le 2^m$. Color white the any square (x, y) for which there exists a positive integer k with $x \equiv y \equiv 2^{k-1} \pmod{2}^k$. Then, color white the square $(0, 0)$.

Color the remaining squares black. Some calculations show that this is a valid construction which achieves $\frac{2}{3}n(n+1)-1$. ■

This second solution was suggested by Kevin Sun.

Solution 3. We can achieve the bound of $\frac{2}{3}n(n+1)-1$ as above. We will now give a construction which works for all $n = 6k+5$. Let $M = 3k+2$.



Consider the board as points (x, y) where $-M \le x, y \le M$. Paint white the following types of squares:

- The origin $(0, 0)$ and the corner (M, M).
- Squares of the form $(\pm a, 0)$ and $(0, \pm a)$, where $a \not\equiv 1 \pmod 3$ and $0 < a < M$.
- Any square $(\pm x, \pm y)$ such that $y - x \equiv 0 \pmod 3$ and $0 < x, y < M$.

Paint black the remaining squares. This yields the desired construction. ■

This third solution was suggested by Ashwin Sah.

C3

We say a finite set S of points in the plane is *very* if for every point X in S, there exists an inversion with center X mapping every point in S other than X to another point in S (possibly the same point).

(a) Fix an integer n. Prove that if $n \geq 2$, then any line segment $\overline{AB}$ contains a unique very set S of size n such that $A, B \in S$.

(b) Find the largest possible size of a very set not contained in any line.

(Here, an *inversion* with center O and radius r sends every point P other than O to the point P' along ray OP such that $OP \cdot OP' = r^2$.)

Sammy Luo

Answer. For part (b), the maximal size is 5.

Solution. For part (a), take a regular $(n+1)$-gon and number the vertices A_i $(i = 0, 1, 2, \ldots, n)$ Now invert the polygon with center A_0 with arbitrary power. This gives a very set of size n. (This can be easy checked with angle chase, PoP, etc.) By scaling and translation, this shows the existence of a very set as in part (a).

It remains to prove uniqueness. Suppose points $A = P_1, P_2, \ldots, P_n = B$ and $A = X_1, X_2, \ldots, X_n = B$ are two very sets on $\overline{AB}$ in that order. Assume without loss of generality that $X_1X_2 > P_1P_2$. Then $X_2X_1^2 = X_2X_3 \cdot (X_1X_n - X_1X_2) \implies X_2X_3 > P_2P_3$. Proceeding inductively, we find $X_kX_{k+1} > P_kP_{k+1}$ for $k = 1, 2, \ldots, n-1$. Thus, $X_1X_n > P_1P_n$, which is a contradiction.

For (b), let $P(A)$ (let's call this power, A is a point in space) be a function returning the radius of inversion with center A. Note that the power of endpoints of 1D very sets are equal, and these powers are the highest out of all points in the very set. Let the convex hull of our very set be H. Let the vertices be $A_1, A_2, ..., A_m$. (We have $m \geq 3$ since the points are not collinear.) Since A_1, A_2 are endpoints of a 1D very set, they have equal power. Going around the hull, all vertices have equal power.

Lemma 1. *Other than the vertices, no other points lie on the edges of H, and H is equilateral.*

Proof. Say X is on A_1A_2. Then X, A_3 are on opposite ends of a 1D very set, so they have equal power. Then $P(X) = P(A_1) = P(A_2)$ contradicting the fact the endpoints have the unique highest power. Therefore, since all sides only have 2 points on them, and all vertices have equal power, all sides are equal. □

Lemma 2. *H is a regular polygon.*

Proof. Let's look at the segment A_1A_3. Say that on it we have a very set of size $k-1$. By uniqueness and the construction in (a), and the fact that $P(A_1) = P(A_2) = P(A_3)$, we get that A_1, A_2, A_3 are 3 vertices of a regular k-gon. Now the very set on segment A_1A_3 under inversion at A_2 would map to a regular k-gon. So all vertices of this regular k-gon would be in our set. Assuming that not all angles are equal taking the largest angle who is adjacent to a smaller angle, we contradict convexity. So all angles are equal. Combining this with Lemma 1, H is a regular polygon. □

Lemma 3. *H cannot have more than 4 vertices.*

Proof. Firstly, note that no points can be strictly any of the triangles $A_iA_{i+1}A_{i+2}$. (*) Or else, inverting with center A_{i+1} we get a point outside H. First, let's do if m (number of vertices) is odd. Let $m = 2k+1$. $(k \geq 2)$ Look at the inversive image of A_{2k+1} under inversion with center A_2. Say it maps to X. Note that $P(X) < P(A_i)$ for any i. Now look at the line $A_{k+2}X$. Since A_{k+2} is an endpoint, but $P(X) < P(A_{k+2})$, the other endpoint of this 1D very set must be on ray $A_{k+2}X$ past X, contradicting (*), since no other vertices of H are on this ray. Similarly for m even and ≥ 6 we can also find 2 points like these who contain no other vertices in H on the line through them. □

Lemma 4. *We only have 2 distinct very sets in 2D (up to scaling), an equilateral triangle (when $n = 3$) and a square with its center (when $n = 5$).*

Proof. First if H has 3 points, then by (*) in Lemma 3, no other points can lie inside H. So we get an equilateral triangle. If H has 4 points, then by (*) in Lemma 3, the only other point that we can add into our set is the center of the square. This also must be added, and this gives a very set of size 5. □

Hence, the maximal size is 5. ■

This problem was proposed by Sammy Luo. This solution was given by Yang Liu.

C4

Let r and b be positive integers. The game of *Monis*, a variant of Tetris, consists of a single column of red and blue blocks. If two blocks of the same color ever touch each other, they both vanish immediately. A red block falls onto the top of the column exactly once every r years, while a blue block falls exactly once every b years,

(a) Suppose that r and b are odd, and moreover the cycles are offset in such a way that no two blocks ever fall at exactly the same time. Consider a period of rb years in which the column is initially empty. Determine, in terms of r and b, the number of blocks in the column at the end.

(b) Now suppose r and b are relatively prime and $r + b$ is odd. At time $t = 0$, the column is initially empty. Suppose a red block falls at times $t = r, 2r, \dots, (b-1)r$ years, while a blue block falls at times $t = b, 2b, \dots, (r-1)b$ years. Prove that at time $t = rb$, the number of blocks in the column is $|1 + 2(r-1)(b+r) - 8S|$, where

$$S = \left\lfloor \frac{2r}{r+b} \right\rfloor + \left\lfloor \frac{4r}{r+b} \right\rfloor + \dots + \left\lfloor \frac{(r+b-1)r}{r+b} \right\rfloor .$$

Sammy Luo

Remark. The second part of this problem was suggested by Allen Liu.

Answer. The answer is $2\gcd(r, b)$.

Solution 1. Consider strings of letters x, y, cancelling xx, Here yy. x, y correspond to red, blue blocks, respectively. I'll denote a way for the blocks to fall by (r, b, C), so r is the years between cycle of red blocks, b is cycle between blue blocks, and C is the cycle offset, more specifically how many years after the first red block falls does the first blue block fall. $C < 0$ is possible, that just means that the first blue block falls earlier than the first red block. To do this, we induct on $r + b$. Assume, $gcd(r, b) = 1$.

Now, let $r > b$ and $r = bk + q$, $0 \le q < b$. We have 2 similar cases to consider:

Case 1: q is odd. First we'll do if $C > 0$, and then by the problem statement, $C < b$. We'll actually show that this falling situation is the same as $(q, b, C) = (b, q, -C)$, and then we'll finish this case by induction. In this case, it's easy to see that the falling will result in a sequence like

$$x(y \dots y)x(y \dots y) \dots x(y \dots y).$$

Note that the $(y \dots y)$ each have length either k or $k+1$, with exactly q of those strings having length $k+1$ and the other $b - q$ having length k. Note that k is even. Now for each of the $(y \dots y)$ strings, reduce them to a single letter depending on parity. Now we are left with q y's and still b x's. We show the resultant string is equal to (q, b, C).

This is actually pretty clear using simple remainder arguments. Say that the first x block fell at time 0. Just note that the length of (some y) was $k + 1$ iff the first y in the string of (some y) fell at time t and $0 < t \pmod r < q$ (then $t + kb < kb + q = r$, so another x would still have not appeared, but will appear next). So seeing all this, my claim becomes equivalent to the following assertion: Let l be the smallest positive integer such that $0 < (C + l \cdot b) \pmod r < q$. Let $t = (C + l \cdot b) \pmod r$ Let $j = \frac{(C + l \cdot b) - t}{r}$. Then j is also the smallest positive integer such that $(j + 1) \cdot q > C$. The proof of this is pretty silly. Then $jr + t = C + l \cdot b$. Taking $\pmod b$ gives $C \equiv jq + t$, and since $0 < C < b$, $C \le jq + t < q(j + 1)$. The converse follows from the fact that for anytime the 2 sides match $\pmod b$, we can solve for l. Why is it equivalent? Well, the first time $k+1$ y's appear consecutively in the initial sequence is when $0 < t \pmod r < q$, and the first time (since $k + 1$ is odd) a y would appear in the reduced sequence is when $q(j + 1) > C$. And these match! For the rest, just rotate the sequence and keep going. Now induction gives that it reduces to the string xy or yx.

Ok, now $C < 0$. So then our sequence would be $yyxyyyyxyyyxyy$ or something like that. What we do is the following: We rotate it by putting stuff on the back end, and then use the case $C > 0$, and associativity of cancellation:

$$\begin{aligned} yyxyyyyxyyyxyy &= (yyx)(xyy)yyxyyyyxyyyxyy \\ &= (yyx)(xyyyyxyyyyxyyy)(xyy) \\ &= (yyx)(xy)(xyy) \\ &= yx. \end{aligned}$$

(Computations show that it always ends up this way). So $C < 0$ is finished.

Case 2: q is even. Similar remainder arguments as above show that if $C > 0$, As above, it's equivalent to saying the minimal j with $(b-q)(j+1) > b-C$ is also the minimal j with $C+l\cdot b = j\cdot r+t$ and $q < t < b$. Taking $\pmod b$, we get $b-C \equiv j(b-q)-t$. But $0 < -t \pmod b < b-q$. So $b-C \le j(b-q)+(b-q) = (j+1)(b-q)$, as desired. ■

This first solution was suggested by Yang Liu.

Solution 2. As in Yang's solution have (r, b, C) represent the state. WLOG $r > b$ so we can set $0 < C < b$. Only $\lfloor C \rfloor$ actually matters so there are b possibilities. Before deletion, the sequence consists of b blue blocks in a cycle with some number of red blocks between each adjacent pair. We can see that taking any possible sequence and shifting the numbers of red blocks between each pair right one pair gives an equivalent sequence, but since $(r, b) = 1$ all of these are distinct, so they're the only possibilities.

So now every (r, b, C) is equivalent to (r, b, ϵ) where $0 < \epsilon < 1$, except shifted. Basically this yields xyS, where S is what would have resulted from all the nonsimultaneous blocks if we allowed $C = 0$. But by symmetry S is symmetric about its center $\frac{rb}{2}$, so everything cancels out in pairs from the center outwards, until we're left with xy.

Basically this leaves the issue of what the offset, in changing the point at which the cyclic sequence's wrap-over is broken, does. Let the unshifted string be $xySA$, where A is the part that is cut off and shifted to the left. Since SA must be a palindrome by the symmetry argument above, S is of the form $(A^{-1})(S')$, where A^{-1} is A in reverse and S' is a palindrome. Then the shifted string cancels to $AxyA^{-1}$. We claim this cancels with only two elements remaining. Indeed we can keep reducing the size of the A; since A's last element is the same as A^{-1}'s first, one of them has to cancel with one of x, y, leaving $A'yxA'^{-1}$, where $|A'| = |A| - 1$, and this continues until only xy or yx remains. ■

This second solution was suggested by Allen Liu.

This problem was proposed by Sammy Luo.

C6

Let f_0 be the function from $\mathbb{Z}^2$ to $\{0,1\}$ such that $f_0(0,0)=1$ and $f_0(x,y)=0$ otherwise. For each positive integer m, let $f_m(x,y)$ be the remainder when

$$f_{m-1}(x,y)+\sum_{j=-1}^{1}\sum_{k=-1}^{1} f_{m-1}(x+j,y+k)$$

is divided by 2. Finally, for each nonnegative integer n, let a_n denote the number of pairs (x,y) such that $f_n(x,y)=1$. Find a closed form for a_n.

Bobby Shen

Solution. Note that a_i is simply the number of odd coefficients of $A_i(x,y)=A(x,y)^i$, where $A(x,y)=(x^2+x+1)(y^2+y+1)-xy$. Throughout this proof, we work in $\mathbb{F}_2$ and repeatedly make use of the Frobenius endomorphism in the form $A_{2^k m}(x,y)=A_m(x,y)^{2^k}=A_m(x^{2^k},y^{2^k})$ (*). We advise the reader to try the following simpler problem before proceeding: "Find (a recursion for) the number of odd coefficients of $(x^2+x+1)^{2^n-1}$."

First suppose n is not of the form 2^m-1, and has $i\ge 0$ ones before its first zero from the right. By direct exponent analysis (after using (*)), we obtain $a_n=a_{\frac{n-(2^i-1)}{2}}a_{2^i-1}$. Applying this fact repeatedly, we find that $a_n=a_{2^{\ell_1}-1}\cdots a_{2^{\ell_r}-1}$, where $\ell_1,\ell_2,\dots,\ell_r$ are the lengths of the r consecutive strings of ones in the binary representation of n. (When $n=2^m-1$, this is trivially true. When $n=0$, we take $r=0$ and a_0 to be the empty product 1, by convention.)

We now restrict our attention to the case $n=2^m-1$. The key is to look at the exponents of x and y modulo 2 – in particular, $A_{2n}(x,y)=A_n(x^2,y^2)$ has only "$(0,0) \pmod 2$" terms for $i\ge 1$. This will allow us to find a recursion.

For convenience, let $U[B(x,y)]$ be the number of odd coefficients of $B(x,y)$, so $U[A_{2^n-1}(x,y)]=a_{2^n-1}$. Observe that

$$\begin{aligned}
A(x,y)&=(x^2+x+1)(y^2+y+1)-xy=(x^2+1)(y^2+1)+(x^2+1)y+x(y^2+1)\\
(x+1)A(x,y)&=(y^2+1)+(x^2+1)y+x^3(y^2+1)+(x^3+x)y\\
(x+1)(y+1)A(x,y)&=(x^2y^2+1)+(x^2y+y^3)+(x^3+xy^2)+(x^3y^3+xy)\\
(x+y)A(x,y)&=(x^2+y^2)+(x^2+1)(y^3+y)+(x^3+x)(y^2+1)+(x^3y+xy^3).
\end{aligned}$$

Hence for $n\ge 1$, we have (using (*) again)

$$\begin{aligned}
U[A_{2^n-1}(x,y)]&=U[A(x,y)A_{2^{n-1}-1}(x^2,y^2)]\\
&=U[(x+1)(y+1)A_{2^{n-1}-1}(x,y)]+U[(y+1)A_{2^{n-1}-1}(x,y)]+U[(x+1)A_{2^{n-1}-1}(x,y)]\\
&=U[(x+1)(y+1)A_{2^{n-1}-1}(x,y)]+2U[(x+1)A_{2^{n-1}-1}(x,y)].
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
U[(x+1)A_{2^n-1}]&=2U[(y+1)A_{2^{n-1}-1}]+2U[(x+1)A_{2^{n-1}-1}]=4U[(x+1)A_{2^{n-1}-1}]\\
U[(x+1)(y+1)A_{2^n-1}]&=2U[(xy+1)A_{2^{n-1}-1}]+2U[(x+y)A_{2^{n-1}-1}]=4U[(x+y)A_{2^{n-1}-1}]\\
U[(x+y)A_{2^n-1}]&=2U[(x+1)(y+1)A_{2^{n-1}-1}]+2U[(x+y)A_{2^{n-1}-1}].
\end{aligned}$$

Here we use the symmetry between x and y, and the identity $(xy+1)=y(x+y^{-1})$.) It immediately follows that

$$\begin{aligned}
U[(x+1)(y+1)A_{2^{n+1}-1}]&=4U[(x+y)A_{2^n-1}]\\
&=8U[(x+1)(y+1)A_{2^{n-1}-1}]+8\frac{U[(x+1)(y+1)A_{2^n-1}]}{4}
\end{aligned}$$

for all $n \geq 1$, and because $x - 4 \mid (x+2)(x-4) = x^2 - 2x - 8$,

$$U[A_{2^{n+2}-1}(x,y)] = 2U[A_{2^{n+1}-1}(x,y)] + 8U[A_{2^n-1}(x,y)]$$

as well. But $U[A_{2^0-1}] = 1$, $U[A_{2^1-1}] = 8$, and

$$U[A_{2^2-1}] = 4U[x+y] + 8U[x+1] = 24,$$

so the recurrence also holds for $n = 0$. Solving, we obtain $a_{2^n-1} = \frac{5\cdot 4^n - 2(-2)^n}{3}$, so we're done. ■

This problem and solution were proposed by Bobby Shen.

Remark. The number of odd coefficients of $(x^2+x+1)^n$ is the Jacobsthal sequence (OEIS A001045) (up to translation). The sequence $\{a_n\}$ in the problem also has a (rather empty) OEIS entry. It may be interesting to investigate the generalization

$$\sum_{j=-1}^{1}\sum_{k=-1}^{1} c_{j,k} f_{i-1}(x+j, y+k)$$

for 9-tuples $(c_{j,k}) \in \{0,1\}^9$. Note that when all $c_{j,k}$ are equal to 1, we get $(x^2+x+1)^n(y^2+y+1)^n$, and thus the square of the Jacobsthal sequence.

Even more generally, one may ask the following: "Let f be an integer-coefficient polynomial in $n \geq 1$ variables, and p be a prime. For $i \geq 0$, let a_i denote the number of nonzero coefficients of f^{p^i-1} (in $\mathbb{F}_p$).

Under what conditions must there always exist an infinite arithmetic progression AP of positive integers for which $\{a_i : i \in AP\}$ satisfies a linear recurrence?"

G1

Let ABC be a triangle with symmedian point K. Select a point A_1 on line BC such that the lines AB, AC, A_1K and BC are the sides of a cyclic quadrilateral. Define B_1 and C_1 similarly. Prove that A_1, B_1, and C_1 are collinear.

Sammy Luo

Solution 1. Let KA_1 intersect AC, AB at A_b, A_c respectively, and analogously define the points B_c, B_a, C_a, C_b. We claim that $A_bA_cB_cB_aC_aC_b$ is cyclic with center K. It's well known that $KA_b = KA_c$, etc. due to the antiparallelisms. Now note $\angle B_cA_cK = \angle AA_cA_b = \angle BCA = \angle B_aB_cB = \angle KB_cA_c$ so we also have $KA_c = KB_c$, etc. So all six segments from K are equal. Now Pascal on $A_bA_cB_cB_aC_aC_b$ gives A_1, B_1, C_1 collinear as wanted. ■

This problem and solution were proposed by Sammy Luo.

Solution 2. Let DEF be the triangle formed by the tangents to the circumcircle of ABC at A, B, and C. Let A', B', C' be $EF \cap BC$, $DF \cap AC$, and $DE \cap AB$, respectively. Since EF is a tangent, it is antiparallel to BC through A, so $A_1K \parallel EF$. Then $A_1B = A_1K \cdot \frac{A'B}{A'E}$, and $A_1C = A_1K \cdot \frac{A'C}{A'F}$ by similar triangles, so

$$\begin{aligned}\frac{A_1B}{A_1C}\frac{B_1C}{B_1A}\frac{C_1A}{C_1B} &= \frac{A'B \cdot A'F}{A'C \cdot A'E} \cdot \frac{B'C \cdot B'D}{B'A \cdot B'F} \cdot \frac{C'A \cdot C'E}{C'B \cdot C'D} \\ &= \frac{BA'}{A'C}\frac{CB'}{B'A}\frac{AC'}{C'B} \cdot \frac{FA'}{A'E}\frac{EC'}{C'D}\frac{DB'}{B'F} \\ &= 1 \cdot 1 \\ &= 1\end{aligned}$$

by Menelaus. (DEF is collinear, since it is the symmedian line) Thus by the converse of Menelaus, A_1, B_1, and C_1 are collinear. ■

This second solution was suggested by Kevin Sun.

G2

$ABCD$ is a cyclic quadrilateral inscribed in the circle ω. Let $AB \cap CD = E$, $AD \cap BC = F$. Let ω_1, ω_2 be the circumcircles of AEF, CEF, respectively. Let $\omega \cap \omega_1 = G$, $\omega \cap \omega_2 = H$. Show that AC, BD, GH are concurrent.

Yang Liu

Solution. Let $AC \cap BD = Q$, $AC \cap GH = Q'$ (assuming $Q \neq Q'$), and let the radical center of ω, ω_1, and ω_2 be P, so P is the intersection of EF, AG, and HC. By Brokard's on $ABCD$, FQE is self-polar, so P (on EF) is on the polar of Q. Similarly, by Brokard's on $AGCH$, Q' is on the polar of P. Thus QQ' is the polar of P, so AC is the polar of P, which is clearly absurd. ■

This problem and solution were proposed by Yang Liu.

G3

Let $A_1A_2A_3\cdots A_{2013}$ be a cyclic 2013-gon. Prove that for every point P not the circumcenter of the 2013-gon, there exists a point $Q \neq P$ such that $\frac{A_iP}{A_iQ}$ is constant for $i \in \{1, 2, 3, \cdots, 2013\}$.

Robin Park

Solution. Let ω be the circumcircle of $A_1A_2A_3\cdots A_{2013}$. We just need Q such that ω is the Apollonius circle of P, Q for some ratio r. Let the center of ω be O, and let PO intersect ω at X, Y. Pick point Q on line XY such that $\frac{XP}{XQ} = \frac{YP}{YQ}$, i.e. $XPYQ$ is harmonic. Now, ω is a circle with center on PQ that has two points X, Y with the same ratio of distances to P, Q, so ω is an Apollonius circle of P, Q; the ratio of distances from any point on ω to P, Q is constant, implying the problem. ■

This problem was proposed by Robin Park. This solution was given by Sammy Luo.

G4

Let $ABCD$ be a quadrilateral inscribed in circle ω. Define $E = AA \cap CD$, $F = AA \cap BC$, $G = BE \cap \omega$, $H = BE \cap AD$, $I = DF \cap \omega$, and $J = DF \cap AB$. Prove that GI, HJ, and the B-symmedian are concurrent.

Robin Park

Solution. The main point of this problem is to show that $AICG$ is harmonic. Indeed, because of similar triangles and the Law of Sines, $AI = \frac{AD \cdot FI}{AF}$ and $CI = 2R\sin(\angle FBI) = 2R \cdot \frac{FI}{FB} \cdot \sin(\angle BID) = \frac{FI \cdot BD}{BF}$. So

$$\frac{AI}{CI} = \frac{AD}{BD} \cdot \frac{BF}{AB} = \frac{AD \cdot AB}{BD \cdot AC} = \frac{AG}{CG}$$

since it's symmetric in B, D.

Therefore, $AICG$ is harmonic. Let $AA \cap CC = K$. Note that I, G, K are collinear. By Pascal's Theorem on $AABGID$, we get that K, H, J are collinear. By the Symmedian Lemma, the B-symmedian passes through K, so HJ, IG, and the B-symmedian all pass through K ■

This problem was proposed by Robin Park. This solution was given by Yang Liu.

G5

Let P be a point in the interior of an acute triangle ABC, and let Q be its isogonal conjugate. Denote by ω_P and ω_Q the circumcircles of triangles BPC and BQC, respectively. Suppose the circle with diameter $\overline{AP}$ intersects ω_P again at M, and line AM intersects ω_P again at X. Similarly, suppose the circle with diameter $\overline{AQ}$ intersects ω_Q again at N, and line AN intersects ω_Q again at Y.

Prove that lines MN and XY are parallel. (Here, the points P and Q are *isogonal conjugates* with respect to $\triangle ABC$ if the internal angle bisectors of $\angle BAC$, $\angle CBA$, and $\angle ACB$ also bisect the angles $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$, respectively. For example, the orthocenter is the isogonal conjugate of the circumcenter.)

Sammy Luo

Solution. We are given that P and Q are isogonal conjugates.

Since $\angle PMX = \angle QNY = \frac{\pi}{2}$, we derive

$$\angle PBX = \angle QBY = \angle PCX = \angle QCY = \frac{\pi}{2}.$$

Thus

$$\angle ABY = \frac{\pi}{2} + \angle ABQ = \angle PBC + \frac{\pi}{2} = \pi - \angle CBX,$$

so X and Y are isogonal with respect to $\angle B$. However, similar angle chasing gives that they are isogonal with respect to $\angle C$. Thus they are isogonal conjugates with respect to ABC. (In particular, $\angle BAY = \angle XAC$.)

Also, $\angle ABY = \pi - \angle CBX = \pi - \angle CMX = \angle AMC$; hence $\triangle ABY \sim \triangle AMC$. Similarly, $\triangle ABN \sim \triangle AXC$. Thus $\frac{AN}{AB} = \frac{AC}{AX}$, and $\frac{AB}{AY} = \frac{AM}{AC}$. Multiplying, we get that $\frac{AN}{AY} = \frac{AM}{AX}$ which implies the conclusion. ■

This problem was proposed by Sammy Luo. This solution was given by Kevin Sun.

Remark. The points M and N are also isogonal conjugates.

G6

Let $ABCD$ be a cyclic quadrilateral with center O. Suppose the circumcircles of triangles AOB and COD meet again at G, while the circumcircles of triangles AOD and BOC meet again at H. Let ω_1 denote the circle passing through G as well as the feet of the perpendiculars from G to AB and CD. Define ω_2 analogously as the circle passing through H and the feet of the perpendiculars from H to BC and DA. Show that the midpoint of GH lies on the radical axis of ω_1 and ω_2.

Yang Liu

Solution 1. Let $F = AB \cap CD$, $E = AD \cap BC$. Let P be the intersection of the diagonals of the quadrilateral ($AC \cap BD$) Then simple angle chasing gives that $APGD$ is cyclic. (Just show that $\angle APD = \angle AGD = \angle AGO + \angle DGO$, both which are easy to find).

Similarly, $BPGC$ is cyclic. Now we show that $\angle PGO = \angle PGA + \angle OGA = \angle PDA + \angle OBA = \pi/2$.

Now by Radical Axis on $BPGC, APGD, ABCD$, we get that E, P, G are collinear. By Radical Axis on $ABGO, CDGO, ABCD$, we get that F, O, G are collinear. Therefore, $\angle EGF = \pi - \angle PGO = \pi/2$. Similarly, $\angle EHF = \pi/2$. So $EFGH$ is cyclic. Similarly, O, H, E are collinear.

Now, the finish is easy. Let M be the midpoint of GH. And let line MGH hit ω_1 at G', and ω_2 at H'. Note that $\angle EH'H = \pi/2 = \angle EGF$, and $\angle EHH' = \angle EFG$. So $\triangle EH'H \sim \triangle EGF \implies HH' = \frac{EH \cdot GF}{EF} = GG'$ by symmetry. So $MH \cdot MH' = MH \cdot (MH + HH') = MG \cdot (MG + GG') = MG \cdot MG'$, so M has the same power wrt both circles, so it's on the radical axis. ■

This problem and solution were proposed by Yang Liu.

Solution 2. Let $P = AB \cap CD, Q = AD \cap BC, R = AC \cap BD$. It's easy to show by angle chasing that the Miquel point M of a cyclic $ABCD$ with center O lies on (AOC). So G, H are the Miquel points of $ACBD, ABDC$ respectively. It's also well-known (by Brokard and a spiral similarity, see here) that G, H are then the feet of the altitudes from O to QR, RP respectively (and O is the orthocenter of PQR).

Note that ω_1, ω_2 are the circles with diameters GP, HQ respectively (due to the right angles). Now, $PQGH$ is cyclic due to the right angles, so the radical center of $(PQGH), \omega_1, \omega_2$ is $GP \cap HQ = O$. Let F be the midpoint of PQ, M the midpoint of GH, and O_1, O_2 the centers of ω_1, ω_2 respectively (so, the midpoints of PG, QH respectively). Now it suffices to show that $OM \perp O_1O_2$. But notice that O_1, O_2 are the feet of perpendiculars from F to PG, QH respectively, and so the line through O that is perpendicular to O_1O_2 is isogonal to OF w.r.t. angle POQ. But since $GHPQ$ is cyclic, GH, PQ are antiparallel wrt this angle, so since OM bisects segment GH, OM is the O-symmedian in $\triangle POQ$, and so is isogonal to OF, and thus perpendicular to O_1O_2 as wanted. So M is on the radical axis as wanted. ■

This second solution was suggested by Sammy Luo.

G7

Let ABC be a triangle inscribed in circle ω with center O; let ω_A be its A-mixtilinear incircle, ω_B be its B-mixtilinear incircle, ω_C be its C-mixtilinear incircle, and X be the radical center of $\omega_A, \omega_B, \omega_C$. Let A', B', C' be the points at which ω_A, ω_B, ω_C are tangent to ω. Prove that AA', BB', CC' and OX are concurrent.

Robin Park

Solution. Let the incenter be I, and the tangency points of the incircle to the 3 sides be T_A, T_B, T_C. Also, let ω_A be tangent to the sides AB, AC at A_B, A_C, respectively (and similar for the other circles and sides). Let the midpoints of the arcs be M_A, M_B, M_C, and the midpoints of T_A, I be N_A, etc.

It's pretty well-known that I is the midpoint of A_B, A_C, and similar. Now we show that the radical axis of ω_B, ω_C contains N_A and M_A. First we show that N_A is on the radical axis. Let (X, ω) denote the power of a point X w.r.t. some circle ω. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function such that $f(P) = (P, \omega_B) - (P, \omega_C)$. Then $f(I) = -IB_C^2 + IC_B^2$ and $f(T_A) = T_AB_C^2 - T_AC_B^2$, so it follows by Pythagorean Theorem that

$$f(I) + f(T_A) = (IC_B^2 - T_AC_B^2) - (IB_C^2 - T_AB_C^2) = IT_A^2 - IT_A^2 = 0.$$

Since f is linear in P, we have that $f(N_A) = \frac{f(I)+f(T_A)}{2} = 0$. Hence N_A lies on the radical axis of ω_B and ω_C.

Now we show that M_A lies on the radical axis. Let l_B be the length of the tangent from M_A to the circle ω_B. By Casey's Theorem on the circles B, M_A, C, ω_B, we get that

$$BM_A \cdot CB_C + CM_A \cdot BB_C = l_B \cdot BC \implies l_B = BM_A = CM_A$$

. Similarly, $l_C = BM_A = CM_A$ (tangent from M_A to ω_C), so M_A lies on their radical axis. Now by simple angle chasing, $M_AM_B \parallel N_AN_B$, so the triangles $M_AM_BM_C$ and $N_AN_BN_C$ are homothetic, so M_AN_A, M_BN_B, M_CN_C are concurrent on IO (the lines through their centers). ■

This problem was proposed by Robin Park. This solution was given by Yang Liu and Robin Park.

G8

In triangle ABC with incenter I and circumcenter O, let A', B', C' be the points of tangency of its circumcircle with its A, B, C-mixtilinear circles, respectively. Let ω_A be the circle through A' that is tangent to AI at I, and define ω_B, ω_C similarly. Prove that $\omega_A, \omega_B, \omega_C$ have a common point X other than I, and that $\angle AXO = \angle OXA'$.

Sammy Luo

Solution. For the sake of simplicity, let D, E, and F be the points of tangency of the circumcircle to the mixtilinear incircles.

Invert with respect to the incircle; $\triangle ABC$ is mapped to $\triangle A'B'C'$. Since the circumcircles of $\triangle A'B'I$, $\triangle B'C'I$, and $\triangle C'A'I$ concur at I, by a well-known lemma I is the orthocenter of $A'B'C'$. Let D', etc. be the images of D, etc., under this inversion. We claim that D' is the reflection of I over the midpoint of $B'C'$. This is clear because A', B', C', and D' are concyclic and ID is a symmedian of $\triangle IBC$, implying that ID' is a median of $\triangle IB'C'$. Therefore D' is also the antipode of A' with respect to the circumcircle of $\triangle A'B'C'$. Similarly, E' and F' are the antipodes of B' and C', respectively.

ω_A is mapped to a line parallel to $A'I$ passing through D', and ω_B, ω_C are mapped similarly. Clearly ω_A', ω_B', and ω_C' concur at the orthocenter of $\triangle D'E'F'$, since $B'C' \parallel E'F'$, $C'A' \parallel F'D'$, and $A'B' \parallel D'E'$. Let this point be X'. Note that $\angle X'A'I = \angle X'D'I$.

We claim that I, X', and O are collinear. If P is the circumcenter of $\triangle A'B'C'$, then note that P is the midpoint of IX' because there exists a homothety centered at O with ratio -1 sending $\triangle A'B'C'$ to $\triangle D'E'F'$ (X' is the de Longchamps point of $\triangle A'B'C'$). Hence O, I, and P are collinear and so it follows that I, X', and O are collinear.

Inverting back to our original diagram, we see that $\angle X'A'I = \angle X'D'I$ implies that $\angle AXO = \angle AXI = \angle IXD = \angle OXD$, as desired. ■

This problem was proposed by Sammy Luo. This solution was given by Robin Park.

G9

Let P be a point inside a triangle ABC such that $\angle PAC = \angle PCB$. Let the projections of P onto BC, CA, and AB be X, Y, Z respectively. Let O be the circumcenter of $\triangle XYZ$, H be the foot of the altitude from B to AC, N be the midpoint of AC, and T be the point such that $TYPO$ is a parallelogram. Show that $\triangle THN$ is similar to $\triangle PBC$.

Sammy Luo

Solution 1. Let Q be the isogonal conjugate of P with respect to ABC. It's well-known that O is the midpoint of PQ. Also, the given angle condition gives $\angle BAQ = \angle PAC = \angle PCB = \angle BCP$, so $\triangle BPC \sim \triangle BQA$. Now let B', P' be the reflections of B, P over AC, respectively, and let T' be the midpoint of QP'. We have $\triangle B'P'C \sim \triangle BPC \sim \triangle BQA$; furthermore, $B'P'C$ and BQA are oriented the same way, so their average (the triangle formed by the midpoints of the segments formed by corresponding points in the triangles), $HT'N$, is directly similar to both of them (for a proof, do some spiral similarity stuff). So it suffices to show $T' = T$. But OYT' is the medial triangle of $P'QP$, so $OT' \parallel PY$ and $YT' \parallel OP$, and so $T' = T$ and we're done. ■

This problem and solution were proposed by Sammy Luo.

Solution 2. Let Q be the reflection of P over O. It's quite well-known and easy to show that Q is the isogonal conjugate of P. Since $\angle PAC = \angle PCB$, $\angle BAQ = \angle PAC = \angle PCB = \angle BCP$. Thus $\triangle BPC \sim \triangle BQA$

Let $S = AP \cap CQ$. Since $\angle CAP = \angle ABQ$, $\triangle CAS$ is isosceles, so $SN \perp AC$. Let P' and Y' are the reflection of P and Y over NS. Since $YP \perp AC \perp NS$, $YPP'Y'$ is a rectangle. Let T' is the reflection of Y over T. Then P, P', Q, and O are the translations of Y, Y', T', and T under vector YP. Thus $Y'T' \parallel P'Q$, so $NT \parallel P'Q$ (since $Y'T'$ is the dilation by 2 from C of NT).

Thus $NT \parallel CQ$, so $\angle HNT = \angle HCQ = \angle PCB$.

Let B' be the reflection of B over PC, and let D be the foot of the perpendicular from B to PC. Then $\triangle B'PC \cong \triangle BPC \sim \triangle BQA$. If we average these triangles, we get that $\triangle BQA \sim \triangle DON$, since D, O, and N, are the midpoints of AC, PQ, and BB' respectively.

Since $NT \parallel CQ$, $\angle HNT = \angle HCQ = \angle PCB = \angle DNO$, so $\angle TNO = \angle HND$.

Now, we know that $\angle CHB = \angle CDB = \frac{\pi}{2}$, so $CHDB$ is cyclic, so $\angle NHD = \angle CHD = \pi - \angle CBD = \pi - (\frac{\pi}{2} - \angle DCB) = \frac{\pi}{2} + \angle PCB = \frac{\pi}{2} + \angle ACQ = \frac{\pi}{2} + \angle ANT = \angle NTO$. Thus $\triangle NHD \sim \triangle NTO$, so $\triangle THN \sim \triangle ODN \sim \triangle QBA \sim \triangle PBC$. ■

This second solution was suggested by Kevin Sun.

G10

We are given triangles ABC and DEF such that $D \in BC, E \in CA, F \in AB$, $AD \perp EF, BE \perp FD, CF \perp DE$. Let the circumcenter of DEF be O, and let the circumcircle of DEF intersect BC, CA, AB again at R, S, T respectively. Prove that the perpendiculars to BC, CA, AB through D, E, F respectively intersect at a point X, and the lines AR, BS, CT intersect at a point Y, such that O, X, Y are collinear.

Sammy Luo

Solution 1. Start with a triangle DEF, circumcircle ω and orthocenter H. Let $DH \cap EF = D_1$, $EH \cap DF = E_1$, $FH \cap DE = F_1$. We already showed that from this a unique triangle ABC. We first show that $HR \perp BC$ and similar stuff. To do this, phantom R', S', T' on $\odot DEF$ so that $HR' \perp R'F$ and similar for S', T'. Let $A' = DR' \cap ES'$, and similar for B', C'. By Radical Axis Theorem on $\odot FT'HF_1, \odot HS'ED_1, \omega$ we get that A', D_1, H are collinear, so $A'D \perp EF$. Since ABC is unique, $R' = R, S' = S, T' = T$. So $HR \perp BC$.

Now we show that $FS \cap ET = K, O, H$ are collinear. For this part we use complex numbers. Let ω be the unit circle. Then $h = d + e + f$. First we find s. s satisfies

$$\frac{s-e}{\overline{s-e}} = -\frac{s-h}{\overline{s-h}}$$

Using $\bar{x} = \frac{1}{x}$ for x on the unit circle, we simplify this to $\frac{s-(d+e+f)}{\frac{1}{s}-\left(\frac{de+df+ef}{def}\right)} = se$, and now we solve for s to find $s = \frac{df(d+f+2e)}{de+ef+2df}$. Now let $K' = OH \cap FS$. Since K' is on OH, we can write it's complex number as $k' = p(d+e+f)$ for a real number p. Now we compute $f - s = f - \frac{df(d+f+2e)}{de+ef+2df} = f\left(1 - \frac{d(d+f+2e)}{de+ef+2df}\right) = f\left(\frac{(f-d)(d+e)}{de+ef+2df}\right)$. Now its pretty easy to compute that $\frac{f-s}{\overline{f-s}} = -\frac{df^2(d+f+2e)}{de+ef+2df}$. So $\frac{k'-f}{\overline{k'-f}} = \frac{f-s}{\overline{f-s}} = -\frac{df^2(d+f+2e)}{de+ef+2df}$. Rearranging, we get

$$k' + \bar{k'} \cdot \frac{df^2(d+f+2e)}{de+ef+2df} = f + \frac{df(d+f+2e)}{de+ef+2df} \implies$$

$$p\left((d+e+f) + \frac{f(d+f+2e)(de+ef+df)}{e(de+ef+2df)}\right) = f\left(\frac{d^2+ef+3de+3df}{de+ef+2df}\right)$$

Now, if we be smart with some manipulation (just use distributive property a lot), we can simplify the above to (after multiplying both sides by $de + ef + 2df$),

$$p\left(\frac{def(d+e+f) + (e+f)(d+e+f)(de+ef+df) + ef(de+ef+df)}{ef}\right) = (d^2+ef+3de+3df)$$

. Now it's easy to see that p will be symmetric in e, f so ET also passes through K'.

Finally, to finish, use Pappus's Theorem on BTF, CSE. Let $BS \cap CT = Y, CF \cap BE = H, FS \cap ET = K$ are collinear. But note that O, H, K are collinear, and that X is the reflection of H over O (since $HR \perp BC$ and similar stuff). So O, X, Y are collinear, as desired. ■

This problem and solution were proposed by Sammy Luo.

Solution 2. This is the same as above, except we will provide a synthetic proof that K, O, and H are collinear. Invert about H. H maps to the incenter of $D'E'F'$. S' is the intersection of the exterior angle bisector of E' with $(D'E'F')$, and T' is defined similarly for F'. Thus S', T' are midpoints of arcs DEF and DFE. We want to prove that H, $K' = (HF'S') \cap (HE'T')$, and the center of $D'E'F'$ are collinear. Let U be the center of this circle and $W = F'S' \cap E'T'$. Since $F'S'E'T'$ is cyclic, W lies on HK', so it suffices to show U, W, H are collinear. Let E_0, F_0 be the other arc midpoints of $D'E', D'F'$. Then Pascal on $DED_0E_0S'T'$ gives U, W, H collinear, so we are done. ■

This second solution was suggested by Michael Kural.

G11

Let ABC be a triangle with circumcenter O. Let P be a point inside ABC, so let the points D, E, F be on BC, AC, AB respectively so that the Miquel point of DEF with respect to ABC is P. Let the reflections of D, E, F over the midpoints of the sides that they lie on be R, S, T. Let the Miquel point of RST with respect to the triangle ABC be Q. Show that $OP = OQ$.

Yang Liu

Solution 1. Let the midpoints of the sides be M_A, M_B, M_C, respectively.

Lemma 1. *Let D, E, F be points on BC, AC, AB respectively. Then there exists a point P such that such that $\angle PFB = \angle PDC = \angle PEA = \alpha$ if and only if*

$$BF^2 + CD^2 + AE^2 = BD^2 + CE^2 + AF^2 + 4K \cot \alpha$$

where K is the area of $\triangle ABC$.

Proof. We apply the Law of Cosines to the triangles $PFB, PFA, PEA, PEC, PDC, PBD$ to get the three equations

$$PF^2 + BF^2 - 2 \cdot PF \cdot BF \cos \alpha = PD^2 + BD^2 + 2 \cdot PD \cdot BD \cos \alpha$$
$$PE^2 + AE^2 - 2 \cdot PE \cdot AE \cos \alpha = PF^2 + AF^2 + 2 \cdot PF \cdot AD \cos \alpha$$
$$PD^2 + CD^2 - 2 \cdot PD \cdot CD \cos \alpha = PE^2 + CE^2 + 2 \cdot PE \cdot CE \cos \alpha$$

Summing this and rearranging terms gives

$$\begin{aligned} BF^2 + AE^2 + CD^2 &= BD^2 + CE^2 + AF^2 \\ &\quad + 2 \cos \alpha \left(PF \cdot BF + PD \cdot BD + PE \cdot AE + PF \cdot AD + PD \cdot CD + PE \cdot CE \right) \\ &= BD^2 + CE^2 + AF^2 + 2 \cos \alpha \cdot \frac{2K}{\sin \alpha} \\ &= BD^2 + CE^2 + AF^2 + 4K \cot \alpha \end{aligned}$$

For the "if" part, just use that if we fix P, D, E, the there is only one point F on AB such that $\angle PFB = \angle PDC = \angle PEA = \alpha$. Also, the equation above only has one solution on the side AB as we move F around. So those 2 points must be the same. □

Lemma 2. *The reflections of PD, PE, PF over M_AO, M_BO, M_CO concur at Q.*

Proof. Since $\angle PFB = \angle PDC = \angle PEA$ (all cyclic quadrilaterals), we can just apply Lemma 1, and do some easy calculations to see that the reflections concur. So let the common intersection point be Q'. Then because opposite angles sum to π, $Q'SCR, Q'TAS, Q'TBR$ all are cyclic, so $Q' = Q$. □

To finish, let $QS \cap PE = Y, QT \cap PF = Z$. By easy angle chasing, $PQYZ$ is cyclic (the points are in some order). Note that $YM_B \cap ZM_C = O$. But also, since YM_B, ZM_C bisect the angles $\angle EYS, \angle FZT$ respectively, the meet at one of the arc midpoints of PQ on the circumcircle of $PQYZ$. So O is the arc midpoint of PQ on the circle $PQYZ$, so $OP = OQ$ as claimed. ■

This problem and solution were proposed by Yang Liu.

Solution 2. Let M_A, M_B, M_C be the midpoints of BC, AC, AB.

I guess we should use directed angles. Let $X = PD \cap QR$, $Y = PE \cap QS$, $Z = PF \cap QT$. Let $\alpha = \angle PDB = \angle PFA = \angle PEC$, and $\beta = \angle CRQ = \angle ASQ = \angle BTQ$. $\angle PXQ = -\angle BDP - \angle QRC = \alpha + \beta$. Similarly, $\angle PYQ = \angle PZQ = \alpha + \beta$. Thus P, Q, X, Y, and Z are concyclic.

Let $G = (AEF) \cap (AST)$, $H = (BFD) \cap (BTR)$, $I = (CDE) \cap (CRS)$. $\angle PGQ = \angle AGQ - \angle AGP = \angle ATQ - \angle AFP = \alpha + \beta$. Similarly, $\angle PHQ = \angle PIQ = \alpha + \beta$, so G, H, and I are on the circle, so P,Q,G,H,I,X,Y,Z are concyclic.

Now, I claim that AG, BH, and CI concur. Consider $\frac{\sin BAG}{\sin GAC} = \frac{\sin FAG}{\sin GAE} = \frac{\sin FEG}{\sin GFE} = \frac{FG}{GE}$. Since $\triangle FGT \sim \triangle EGS$ (due to cyclic quads), $\frac{FG}{GE} = \frac{FT}{ES}$. Thus $\frac{\sin BAG}{\sin GAC}\frac{\sin ACI}{\sin ICB}\frac{\sin CBH}{\sin HBA} = \frac{FT}{ES}\frac{ES}{DR}\frac{DR}{FT} = 1$, so by Ceva's theorem, AG, BH, and CI concur.

Also, since $\triangle FGE \sim \triangle TGS$, spiral similarity gives that $\triangle FGE \sim \triangle TGS \sim M_CGM_B$. Then AM_CGOM_B is cyclic.

Now, let $J = AG \cap BH \cap CI$. Since $\angle AM_CO = \frac{\pi}{2}$, $\angle AGO = \frac{\pi}{2}$, so $\angle JGO = \frac{\pi}{2}$. Similarly, $\angle JHO = \angle JIO = \frac{\pi}{2}$, so J, O, G, H, and I are cyclic with diameter JO. However, from earlier we have that the circumcircle of GHI contains points P, Q, X, Y, Z. Thus $GHIJOPQXYZ$ is a cyclic decagon with diameter OJ.

Then $\angle PDB = \angle PFA = \angle PGA = \angle PGJ = \angle PXJ$, so $BC \parallel JX$. Since OJ is a diameter, $OX \perp XJ$, and since M_A is a midpoint, $OM_A \perp BC$. However, $BC \parallel JX$, so M_A is on OX. However, $DM_A = RM_A$, so $\triangle DM_AX \cong \triangle RM_AX$, so $\angle DXM_A = \angle M_AXR$, so $\angle PXO = \angle OXQ$, so $\angle OPQ = -\angle OQP$, which means that $OP = OQ$. ■

This second solution was suggested by Kevin Sun.

Solution 3. Let AQ meet $APEF$ at L, BQ meet $BPDF$ at K, CQ meet $CPDE$ at G. Let the midpoint of K, Q be M, and the midpoints of the sides by M_A, M_B, M_C. Note that $KDF \sim QRT$ since

$$\angle KDF = \angle KBF = \angle QBT = \angle QRT$$

and similarly $\angle KFD = \angle QTR$, so averaging these two triangles yields another similar triangle MM_AM_C. Then $\angle M_CMM_A = \angle DKF = \pi - \angle DBF$, so BM_CMM_A is cyclic. But clearly this quadrilateral has diameter BO, so $OM \perp BM$. Thus $OQ = OK(= OL = OG)$ by similar arguments. We claim $PKQG$ is cyclic. Indeed,

$$\angle KPG + \angle KQG = 2\pi - \angle KPD - \angle GPD + \angle KQG = \angle BQC + \angle QCB + \angle CBQ = \pi$$

So this quadrilateral is cyclic. Then P lies on cyclic $QKLG$ with center O, so we are done. ■

This third solution was suggested by Michael Kural.

Solution 4. Let A', B', C' be the antipodes of A, B, C, respectively, in $(AEF), (BFD), (CDE)$ respectively; let A'', B'', C'' be the antipodes of A, B, C, respectively, in $(AST), (BTR), (CRS)$, respectively. Now, B', C' are both on the perpendicular to BC through D, and so forth. So note that B', B'' are reflections about O, since the feet from B', B'' to BC, BA are both symmetric about the corresponding midpoints.

Also, note (using directed angles): $\angle PB'B = \angle PFB = \angle PFA = \angle PEA = \angle PA'A = \angle PEC = \angle PDC = \angle PC'C$ and $\angle BPB' = \angle APA' = \angle CPC' = 90^\circ$ so $BB'P, CC'P, AA'P$ are all directly similar; thus P is the center of a spiral similarity (with angle 90°) from $A'B'C'$ to ABC, which we will call S_P. Similarly, Q is the center of a spiral similarity (with angle 90°) from ABC to $A''B''C''$, which we call S_Q.

Now consider the composition S_QS_P (S_P is applied first). This maps $A'B'C'$ to $A''B''C''$. But these two triangles are reflections of each other about O, so O is at the same position relative to both (in fact, it's their center of rotation!); thus S_QS_P maps O to itself. In particular, since $A'B'C', A''B''C''$ are congruent, S_P, S_Q must have scale factors that are multiplicative inverses; say the scale factor of S_P is r.

So let O' be the image of O under S_P. So $OPO' = 90^\circ$ and $O'QO = 90^\circ$; $\frac{O'P}{OP} = r = \frac{O'Q}{OQ}$. This is enough to show OPO', OQO' congruent, so $OP = OQ$ as desired. ■

This fourth solution was suggested by Sammy Luo.

Remark. This is quite similar in flavor to IMO Shortlist 2012, Problem G6, and a comment given by user **proglote** in that thread can be used to solve this problem.

Remark. In fact, a further generalization of this problem of this problem is possible. Let P a point, and XYZ be its pedal triangle. A_1, B_1, and C_1 are points on BC, AC, and AB, and A_2, B_2, and C_2 are their reflections over X, Y, and Z, If the Miquel point of A_1, B_1, C_1 is P_1 and the Miquel point of A_2, B_2, C_2 is P_2, then $PP_1 = PP_2$.

G12

Let $AB = AC$ in $\triangle ABC$, and let D be a point on segment AB. The tangent at D to the circumcircle ω of BCD hits AC at E. The other tangent from E to ω touches it at F, and $G = BF \cap CD$, $H = AG \cap BC$. Prove that $BH = 2HC$.

David Stoner

Solution 1. Let J be the second intersection of ω and AC, and X be the intersection of BF and AC. It's well-known that $DJFC$ is harmonic; perspectivity wrt B implies $AJXC$ is also harmonic. Then $\frac{AJ}{JX} = \frac{AC}{CX} \implies (AJ)(CX) = (AC)(JX)$. This can be rearranged to get

$$(AJ)(CX) = (AJ + JX + XC)(JX) \implies 2(AJ)(CX) = (JX + AJ)(JX + XC) = (AX)(CJ),$$

so

$$\left(\frac{AX}{XC}\right)\left(\frac{CJ}{JA}\right) = 2.$$

But $\frac{CJ}{JA} = \frac{AD}{DB}$, so by Ceva's we have $BH = 2HC$, as desired. ■

Solution 2. Let J be the second intersection of ω and AC. It's well-known that $DJFC$ is harmonic; thus we have $(DJ)(FC) = (JF)(DC)$. By Ptolemy's, this means

$$(DF)(JC) = (DJ)(FC) + (JF)(DC) = 2(JD)(CF) \implies \left(\frac{JC}{JD}\right)\left(\frac{FD}{FC}\right) = 2.$$

Yet $JC = DB$ by symmetry, so this becomes

$$2 = \left(\frac{DB}{JD}\right)\left(\frac{FD}{FC}\right) = \left(\frac{\sin DJB}{\sin JBD}\right)\left(\frac{\sin FCD}{\sin FDC}\right) = \left(\frac{\sin DCB}{\sin ACD}\right)\left(\frac{\sin FBA}{\sin CBF}\right).$$

Thus by (trig) Ceva's we have $\frac{\sin BAH}{\sin CAH} = 2$, and since $AB = AC$ it follows that $BH = 2HC$, as desired. ■

This problem and its solutions were proposed by David Stoner.

G13

Let ABC be a nondegenerate acute triangle with circumcircle ω and let its incircle γ touch AB, AC, BC at X, Y, Z respectively. Let XY hit arcs AB, AC of ω at M, N respectively, and let $P \neq X, Q \neq Y$ be the points on γ such that $MP = MX, NQ = NY$. If I is the center of γ, prove that P, I, Q are collinear if and only if $\angle BAC = 90^\circ$.

David Stoner

Solution. Let α be the half-angles of $\triangle ABC$, r inradius, and u, v, w tangent lengths to the incircle. Let $T = MP \cap NQ$ so that I is the incenter of $\triangle MNT$. Then $\angle IPT = \angle IXY = \alpha = \angle IYX = \angle IQT$ gives $\triangle TIP \sim \triangle TIQ$, so P, I, Q are collinear iff $\angle TIP = 90^\circ$ iff $\angle MTN = 180^\circ - 2\alpha$ iff $\angle MIN = 180^\circ - \alpha$ iff $MI^2 = MX \cdot MN$. First suppose I is the center of γ. Since A, I are symmetric about XY, $\angle MAN = \angle MIN$. But P, I, Q are collinear iff $\angle MIN = 180^\circ - \alpha$, so because arcs AN and BM sum to 90°, P, I, Q are collinear iff arcs BM, MA have the same measure. Let $M' = CI \cap \omega$; then $\angle BM'I = \angle BM'C = 90^\circ - \angle BXI$, so $M'XIBZ$ is cyclic and $\angle M'XB = \angle M'IB = 180^\circ - \angle BIC = 45^\circ = \angle AXY$, as desired. (There are many other ways to finish as well.) Conversely, if P, I, Q are collinear, then by power of a point, $m(m+2t) = MI^2 - r^2 = MX \cdot MN - r^2 = m(m+2t+n) - r^2$, so $mn = r^2$. But we also have $m(n+2t) = uv$ and $n(m+2t) = uw$, so

$$r^2 = mn = \frac{uv - r^2}{2t}\frac{uw - r^2}{2t} = \frac{\frac{uv(u+v)}{u+v+w}}{2r\cos\alpha}\frac{\frac{uw(u+w)}{u+v+w}}{2r\cos\alpha} = \frac{r^2}{4\cos^2\alpha}\frac{(u+v)(u+w)}{vw}.$$

Simplifying using $\cos^2\alpha = \frac{u^2}{u^2+r^2} = \frac{u(u+v+w)}{(u+v)(u+w)}$, we get

$$0 = (u+v)^2(u+w)^2 - 4uvw(u+v+w) = (u(u+v+w) - vw)^2,$$

which clearly implies $(u+v)^2 + (u+w)^2 = (v+w)^2$, as desired. ■

This problem was proposed by David Stoner. This solution was given by Victor Wang.

N1

Does there exist a strictly increasing infinite sequence of perfect squares $a_1, a_2, a_3, ...$ such that for all $k \in \mathbb{Z}^+$ we have that $13^k | a_k + 1$?

Jesse Zhang

Solution. We have that 5 is a solution to $x^2 + 1 = 0 \mod 13$. Now assume that we have a solution x_k to $f(x) = x^2 + 1 = 0 \mod 13^k$. Note that $f'(x) = 2x \neq 0 \mod 13$ clearly, so by Hensel there is a solution x_{k+1} to $f(x) = x^2 + 1 = 0 \mod 13^{k+1}$. Then just add 13^{k+1} to x_{k+1} to make it strictly larger than x_k, and we're done. ■

This problem was proposed by Jesse Zhang. This solution was given by Michael Kural.

N2

Define the Fibanocci sequence recursively by $F_1 = 1$, $F_2 = 1$ and $F_{i+2} = F_i + F_{i+1}$ for all i. Prove that for all integers $b, c > 1$, there exists an integer n such that the sum of the digits of F_n when written in base b is greater than c.

Ryan Alweiss

Solution. It's well known that if N is a positive integer multiple of $b^k - 1$, then the base b digital sum of N is at least $k(b-1)$. Now just apply the lemma with k sufficiently large and pick n with $b^k - 1 \mid F_n$. ■

This problem and solution were proposed by Ryan Alweiss.

N3

Let t and n be fixed integers each at least 2. Find the largest positive integer m for which there exists a polynomial P, of degree n and with rational coefficients, such that the following property holds: exactly one of

$$\frac{P(k)}{t^k} \text{ and } \frac{P(k)}{t^{k+1}}$$

is an integer for each $k = 0, 1, ..., m$.

Michael Kural

Answer. The maximal value of m is n.

Solution 1. Note that if $t^{k+1} \parallel P(k+1)$ and $t^k \parallel P(k)$, then $t^k \parallel P(k+1) - P(k)$. A simple induction on $\deg P$ then establishes an upper bound of n. To achieve this, simply put $P(k) = t^k$ for each $0 \le k \le n$. ■

This problem and solution were proposed by Michael Kural.

Solution 2. By Lagrange Interpolation, we can find a polynomial satisfying $P(k) = t^k$ for $0 \le k \le n$ with rational coefficients. By Newtonian Interpolation, $P(n+1) = \sum_{i=0}^{n} \binom{n}{i} P(i)(-1)^{n-i}$. Taking $\pmod t$, $P(n+1) = (-1)^n \cdot P(0) \neq 0 \pmod t$. ■

This second solution was suggested by Yang Liu.

N4

Let $\mathbb{N}$ denote the set of positive integers, and for a function f, let $f^k(n)$ denote the function f applied k times. Call a function $f : \mathbb{N} \to \mathbb{N}$ *saturated* if

$$f^{f^{f(n)}(n)}(n) = n$$

for every positive integer n. Find all positive integers m for which the following holds: every saturated function f satisfies $f^{2014}(m) = m$.

Evan Chen

Answer. All m dividing 2014; that is, $\{1, 2, 19, 38, 53, 106, 1007, 2014\}$.

Solution. First, it is easy to see that f is both surjective and injective, so f is a permutation of the positive integers. We claim that the functions f which satisfy the property are precisely those functions which satisfy $f^n(n) = n$ for every n.

For each integer n, let $\operatorname{ord}(n)$ denote the smallest integer k such that $f^k(n)$. These orders exist since $f^{f^{f(n)}(n)}(n) = n$, so $\operatorname{ord}(n) \le f^{f(n)}(n)$; in fact we actually have

$$\operatorname{ord}(n) \mid f^{f(n)}(n) \tag{8.1}$$

as a consequence of the division algorithm.

Since f is a permutation, it is immediate that $\operatorname{ord}(n) = \operatorname{ord}(f(n))$ for every n; this implies easily that $\operatorname{ord}(n) = \operatorname{ord}\left(f^k(n)\right)$ for every integer k. In particular, $\operatorname{ord}(n) = \operatorname{ord}\left(f^{f(n)-1}(n)\right)$. But then, applying (8.1) to $f^{f(n)-1}(n)$ gives

$$\begin{aligned}\operatorname{ord}(n) = \operatorname{ord}\left(f^{f(n)-1}(n)\right) &\mid f^{f\left(f^{f(n)-1}(n)\right)}\left(f^{f(n)-1}(n)\right) \\ &= f^{f^{f(n)}(n)+f(n)-1}(n) \\ &= f^{f(n)-1}\left(f^{f^{f(n)}(n)}(n)\right) \\ &= f^{f(n)-1}(n)\end{aligned}$$

Inductively, then, we are able to show that $\operatorname{ord}(n) \mid f^{f(n)-k}(n)$ for every integer k; in particular, $\operatorname{ord}(n) \mid f^0(n) = n$, which implies that $f^n(n) = n$. To see that this is actually sufficient, simply note that $\operatorname{ord}(n) = \operatorname{ord}(f(n)) = \cdots$, which implies that $\operatorname{ord}(n) \mid f^k(n)$ for every k.

In particular, if $m \mid 2014$, then $\operatorname{ord}(m) \mid m \mid 2014$ and $f^{2014}(m) = m$. The construction for the other values of m (showing that they are not forced) is left as an easy exercise. ■

This problem and solution were proposed by Evan Chen.

Remark. There are many ways to express the same ideas. For instance, the following approach ("unraveling indices") also works: It's not hard to show that f is a bijection with finite cycles (when viewed as a permutation). If $C = (n_0, n_1, \ldots, n_{\ell-1})$ is one such cycle with $f(n_i) = n_{i+1}$ for all i (extending indices mod ℓ), then $f^{f^{f(n)}(n)}(n) = n$ holds on C iff $\ell \mid f^{f(n_i)}(n_i) = n_{i+n_{i+1}}$ for all i. But $\ell \mid n_j \implies \ell \mid n_{j-1+n_j} = n_{j-1}$ for fixed j, so the latter condition holds iff $\ell \mid n_i$ for all i. Thus $f^{2014}(n) = n$ is forced unless and only unless $n \nmid 2014$.

N5

Define a *beautiful number* to be an integer of the form a^n, where $a \in \{3, 4, 5, 6\}$ and n is a positive integer. Prove that each integer greater than 2 can be expressed as the sum of pairwise distinct beautiful numbers.

Matthew Babbitt

Solution. First, we prove a lemma.

Lemma 1. *Let $a_0 > a_1 > a_2 > \cdots > a_n$ be positive integers such that $a_0 - a_n < a_1 + a_2 + \cdots + a_n$. Then for some $1 \le i \le n$, we have*

$$0 \le a_0 - (a_1 + a_2 + \cdots + a_i) < a_i.$$

Proof. Proceed by contradiction; suppose the inequalities are all false. Use induction to show that $a_0 - (a_1 + \cdots + a_i) \ge a_i$ for each i. This becomes a contradiction at $i = n$. □ □

Let N be the integer we want to express in this form. We will prove the result by strong induction on N. The base cases will be $3 \le N \le 10 = 6 + 3 + 1$.

Let $x_1 > x_2 > x_3 > x_4$ be the largest powers of $3, 4, 5, 6$ less than $N - 3$, in some order. If one of the inequalities of the form

$$3 \le N - (x_1 + \cdots + x_k) < x_k + 3; \quad 1 \le k \le 4$$

is true, then we are done, since we can subtract of $x_1, \ldots, x_k$ from N to get an N' with $3 \le N' < N$ and then apply the inductive hypothesis; the construction for N' cannot use any of $\{x_1, \ldots, x_k\}$ since $N' - x_k < 3$.

To see that this is indeed the case, first observe that $N - 3 > x_1$ by construction and compute

$$x_1 + x_2 + x_3 + x_4 + x_4 \ge (N - 3) \cdot \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{6} \right) > N - 3.$$

So the hypothesis of the lemma applies with $a_0 = N - 3$ and $a_i = x_i$ for $1 \le i \le 4$.

Thus, we are done by induction. ■

This problem and solution were proposed by Matthew Babbitt.

Remark. While the approach of subtracting off large numbers and inducting is extremely natural, it is not immediately obvious that one should consider $3 \le N - (x_1 + \cdots + x_k) < x_k + 3$ rather than the stronger bound $3 \le N - (x_1 + \cdots + x_k) < x_k$. In particular, the solution method above does not work if one attempts to get the latter.

N6

Show that the numerator of

$$\frac{2^{p-1}}{p+1}-\left(\sum_{k=0}^{p-1}\frac{\binom{p-1}{k}}{(1-kp)^2}\right)$$

is a multiple of p^3 for any odd prime p.

Yang Liu

Solution. Remark $(1-kp)^2(1+2pk+3p^2k^2)\equiv 3k^4p^4-4k^3p^3+1\equiv 1 \pmod{p^3}$, so $\frac{1}{(1-kp)^2}\equiv(1+2pk+3p^2k^2) \pmod{p^3}$. Thus

$$\begin{aligned}
\left(\sum_{k=0}^{p-1}\frac{\binom{p-1}{k}}{(1-kp)^2}\right) &\equiv \sum_{k=0}^{p-1}\binom{p-1}{k}(1+2pk+3p^2k^2) \pmod{p^3}\\
&=\sum_{k=0}^{p-1}\binom{p-1}{k}+\sum_{k=0}^{p-1}2pk\binom{p-1}{k}+\sum_{k=0}^{p-1}3p^2k^2\binom{p-1}{k}\\
&=2^{p-1}+\sum_{k=0}^{p-1}pk\binom{p-1}{k}+\sum_{k=0}^{p-1}p(p-1-k)\binom{p-1}{k}+\sum_{k=0}^{p-1}3p^2k^2\binom{p-1}{k}\\
&=2^{p-1}+\sum_{k=0}^{p-1}p(p-1)\binom{p-1}{k}+\sum_{k=0}^{p-1}3p^2k^2\binom{p-1}{k}\\
&=(p^2-p+1)2^{p-1}+\sum_{k=0}^{p-1}3p^2k^2\binom{p-1}{k}\\
&\equiv(p^2-p+1)2^{p-1}+\sum_{k=0}^{p-1}3p^2k^2(-1)^k \pmod{p^3}\\
&\equiv(p^2-p+1)2^{p-1}+3p^3\frac{p-1}{2} \pmod{p^3}\\
&\equiv\frac{2^{p-1}}{p+1} \pmod{p^3}
\end{aligned}$$

■

This problem and solution were proposed by Yang Liu.

N7

Find all triples (a, b, c) of positive integers such that if n is not divisible by any prime less than 2014, then $n + c$ divides $a^n + b^n + n$.

Evan Chen

Answer. $(a, b, c) = (1, 1, 2)$.

Solution. Let p be an arbitrary prime such that $p \geq 2011 \cdot \max\{abc, 2013\}$. By the Chinese Remainder Theorem it is possible to select an integer n satisfying the following properties:

$$\begin{aligned} n &\equiv -c \pmod{p} \\ n &\equiv -1 \pmod{p-1} \\ n &\equiv -1 \pmod{q} \end{aligned}$$

for all primes $q \leq 2011$ not dividing $p - 1$. This will guarantee that n is not divisible by any integer less than 2013. Upon selecting this n, we find that

$$p \mid n + c \mid a^n + b^n + n$$

which implies that

$$a^n + b^n \equiv c \pmod{p}$$

But $n \equiv -1 \pmod{p-1}$; hence $a^n \equiv a^{-1} \pmod{p}$ by Euler's Little Theorem. Hence we may write

$$p \mid ab(a^{-1} + b^{-1} - c) = a + b - abc.$$

But since p is large, this is only possible if $a + b - abc$ is zero. The only triples of positive integers with that property are $(a, b, c) = (2, 2, 1)$ and $(a, b, c) = (1, 1, 2)$. One can check that of these, only $(a, b, c) = (1, 1, 2)$ is a valid solution. ■

This problem and solution were proposed by Evan Chen.

N8

Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that:

(i) The greatest common divisor of the sequence $f(1), f(2), \dots$ is 1.

(ii) For all sufficiently large integers n, we have $f(n) \neq 1$ and

$$f(a)^n \mid f(a+b)^{a^{n-1}} - f(b)^{a^{n-1}}$$

for all positive integers a and b.

Yang Liu

Answer. The only such function is the constant function $f(b) = b$.

Solution. Let (ii) hold for $n \geq C$. First we claim $f(a) \mid a$ for all a. Let p be any prime dividing $f(a)$. Choose b so that $p \nmid f(a+b), f(b)$ (possible via (i)). So

$$p \mid f(a+b)^{a^{C-1}} - f(b)^{a^{C-1}}.$$

Now let

$$\upsilon_p\left(f(a+b)^{a^{C-1}} - f(b)^{a^{C-1}}\right) = k.$$

By the divisibility for all $n > C$,

$$n\upsilon_p(f(a)) \leq \upsilon_p\left(f(a+b)^{a^{n-1}} - f(b)^{a^{n-1}}\right) = k + (n-C)\upsilon_p(a)$$

by Lifting the Exponent. Now it's clear that $\upsilon_p(f(a)) \leq \upsilon_p(a)$, so $f(a) \mid a$.

Note that for sufficiently large primes p since $f(p) \mid p$, and then $f(p) \neq 1$, $f(p) = p$. Now plug in $a = p$, and by Fermat's Little Theorem, $p \mid f(b+p) - f(b)$ for all b and sufficiently large p. In fact, this then gives that

$$p \mid f(b+kp) - f(b)$$

for any integer k. Now choose $p > b$. If $f(b+p) \neq b+p$, then

$$f(b+p) \leq \frac{b+p}{2} < p.$$

But $p \mid f(b+p) - f(b)$ for all large enough p. Therefore $f(b+p) = f(b)$ for all sufficiently large primes p. By our condition, $f(b) \neq 1$ now, so take a prime $q \mid f(b)$. Then $q \mid b$ and therefore, $q \mid f(b+p) - f(p) = f(b) - f(p) \implies q \mid p$ for any sufficiently large p. So $q = 1$, contradiction. So $f(b+p) = b+p$. Since $0 < f(b+p) - f(b) = b+p-f(b) < b+p < 2p$ and $p \mid f(b+p) - f(b)$, $f(b) = b$ for all b. You can check that this solution works with LTE. ■

This problem and solution were proposed by Yang Liu.

N11

Let p be a prime satisfying $p^2 \mid 2^{p-1} - 1$, and let n be a positive integer. Define

$$f(x) = \frac{(x-1)^{p^n} - (x^{p^n} - 1)}{p(x-1)}.$$

Find the largest positive integer N such that there exist polynomials $g(x)$, $h(x)$ with integer coefficients and an integer r satisfying $f(x) = (x-r)^N g(x) + p \cdot h(x)$.

Victor Wang

Answer. The largest possible N is $2p^{n-1}$.

Solution 1. Let $F(x) = \frac{x}{1} + \cdots + \frac{x^{p-1}}{p-1}$.

By standard methods we can show that $(x-1)^{p^n} - (x^{p^{n-1}} - 1)^p$ has all coefficients divisible by p^2. But $p^2 \mid 2^{p-1} - 1$ means p is odd, so working in $\mathbb{F}_p$, we have

$$(x-1)f(x) = \sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k}(-1)^{k-1}x^{p^{n-1}k} = \sum_{k=1}^{p-1}\binom{p-1}{k-1}(-1)^{k-1}\frac{x^{p^{n-1}k}}{k}$$
$$= \sum_{k=1}^{p-1}\frac{x^{p^{n-1}k}}{k^{p^{n-1}}} = F(x)^{p^{n-1}},$$

where we use Fermat's little theorem, $\binom{p-1}{k-1} \equiv (-1)^{k-1} \pmod p$ for $k = 1, 2, \ldots, p-1$, and the well-known fact that $P(x^p) - P(x)^p$ has all coefficients divisible by p for any polynomial P with integer coefficients.

However, it is easy to verify that $p^2 \mid 2^{p-1} - 1$ if and only if $p \mid F(-1)$, i.e. -1 is a root of F in $\mathbb{F}_p$. Furthermore, $F'(x) = \frac{x^{p-1}-1}{x-1} = (x+1)(x+2)\cdots(x+p-2)$ in $\mathbb{F}_p$, so -1 is a root of F with multiplicity 2; hence $N \geq 2p^{n-1}$. On the other hand, since F' has no double roots, F has no integer roots with multiplicity greater than 2. In particular, $N \leq 2p^{n-1}$ (note that the multiplicity of 1 is in fact $p^{n-1} - 1$, since $F(1) = 0$ by Wolstenholme's theorem but 1 is not a root of F'). ■

This problem and solution were proposed by Victor Wang.

Remark. The rth derivative of a polynomial P evaluated at 1 is simply the coefficient $[(x-1)^r]P$ (i.e. the coefficient of $(x-1)^r$ when P is written as a polynomial in $x-1$) divided by $r!$.

Solution 2. This is asking to find the greatest multiplicity of an integer root of f modulo p; I claim the answer is $2p^{n-1}$.

First, we shift x by 1 and take the negative (since this doesn't change the greatest multiplicity) for convenience, redefining f as $f(x) = \frac{(x+1)^{p^n} - x^{p^n} - 1}{px}$.

Now, we expand this. We can show, by writing out and cancelling, that p^1 fully divides $\binom{p^n}{k}$ only when p^{n-1} divides k; thus, we can ignore all terms except the ones with degree divisible by p^{n-1} (since they still go away when taking it mod p), leaving $f(x) = \frac{1}{px}(\binom{p^n}{p^{n-1}}x^{p^n - p^{n-1}} + \cdots + \binom{p^n}{p^n - p^{n-1}}x^{p^{n-1}})$.

We can also show, by writing out/cancelling, that $\frac{1}{p}\binom{p^n}{kp^{n-1}} = \frac{1}{p}\binom{p}{k}$ modulo p. Simplifying using this, the expression above becomes $f(x) = \frac{1}{px}(\binom{p}{1}x^{p^n - p^{n-1}} + \cdots + \binom{p}{p-1}x^{p^{n-1}}) = \frac{1}{px}((x^{p^{n-1}} + 1)^p - (x^{p^n} + 1))$.

Now, we ignore the $1/x$ for the moment (all it does is reduce the multiplicity of the root at $x = 0$ by 1) and just look at the rest, $P(x) = \frac{1}{p}((x^{p^{n-1}} + 1)^p - (x^{p^n} + 1))$.

Substituting $y = x^{p^{n-1}}$, this becomes $\frac{1}{p}((y+1)^p - (y^p + 1))$; since $\frac{1}{p}\binom{p}{k} = \frac{1}{k}\binom{p-1}{k-1}$, this is equal to $P(x) = \frac{1}{1}\binom{p-1}{0}y^{p-1} + \cdots + \frac{1}{p-1}\binom{p-1}{p-2}y$. (We work mod p now; the ps can be cancelled before modding out.)

We now show that $P(x)$ has no integer roots of multiplicity greater than 2, by considering the root multiplicities of y times its reversal, or $Q(x) = \frac{1}{p-1}\binom{p-1}{p-2}y^{p-1} + \cdots + \frac{1}{1}\binom{p-1}{0}y$.

Note that some polynomial P has a root of multiplicity m at x iff P and its first $m-1$ derivatives all have zeroes at x. (We're using the formal derivatives here - we can prove this algebraically over $\mathbb{Z}$ mod p, if $m < p$.) The derivative of Q is $\binom{p-1}{p-2}y^{p-2} + \cdots + \binom{p-1}{0}$, or $(y+1)^{p-1} - y^{p-1}$, which has as a root every residue except 0 and -1 by Fermat's little theorem; the second derivative is a constant multiple of $(y+1)^{p-2} - y^{p-2}$, which has no integer roots by Fermat's little theorem and unique inverses. Therefore, no integer root of Q has multiplicity greater than 2; we know that the factorization of a polynomial's reverse is just the reverse of its factorization, and integers have inverses mod p, so $P(x)$ doesn't have integer roots of multiplicity greater than 2 either.

Factoring $P(x)$ completely in y (over some extension of $\mathbb{F}_p$), we know that two distinct factors can't share a root; thus, at most 2 factors have any given integer root, and since their degrees (in x) are each p^{n-1}, this means no integer root has multiplicity greater than $2p^{n-1}$.

However, we see that $y = 1$ is a double root of P. This is because plugging in gives $P(1) = \frac{1}{p}((1+1)^p - (1^p + 1)) = \frac{1}{p}(2^p - 2)$; by the condition, p^2 divides $2^p - 2$, so this is zero mod p. Since 1 is its own inverse, it's a root of Q as well, and it's a root of Q's derivative so it's a double root (so $(y-1)^2$ is part of Q's factorization). Reversing, $(y-1)^2$ is part of P's factorization as well.

Applying a well-known fact, $y - 1 = x^{p^{n-1}} - 1 = (x-1)^{p^{n-1}}$ modulo p, so 1 is a root of P with multiplicity $2p^{n-1}$.

Since adding back in the factor of $1/x$ doesn't change this multiplicity, our answer is therefore $2p^{n-1}$. ■

This second solution was suggested by Alex Smith.