**$17^{\text{th}}$ Ex-Lincoln Math Olympiad**

**ELMO 2015**

**Pittsburgh, PA**

**OFFICIAL SOLUTIONS**

1. Define the sequence $a_1 = 2$ and $a_n = 2^{a_{n-1}} + 2$ for all integers $n \geq 2$. Prove that $a_{n-1}$ divides $a_n$ for all integers $n \geq 2$.

   *Proposed by Sam Korsky.*

   **Solution.** We prove by induction that both $a_{n-1} \mid a_n$ and $a_{n-1} - 1 \mid a_n - 1$ are true for all positive integers $n \geq 2$. We have $1 \mid 5$ and $2 \mid 6$ so the base case works.

   For the inductive step $k \to k+1$, note that

   $$a_{k-1} \mid a_k \implies 2^{a_{k-1}} + 1 \mid 2^{a_k} + 1 \implies a_k - 1 \mid a_{k+1} - 1$$

   $$a_{k-1} - 1 \mid a_k - 1 \implies 2^{a_{k-1}} + 2 \mid 2^{a_k} + 2 \implies a_k \mid a_{k+1}$$

   So the induction is complete and the result follows. (The above works because $x + 1 \mid x^k + 1$ for odd $k$, and $2 \mid a_n$ but $4 \nmid a_n$ for all $n$.) ■

   This problem and solution were proposed by Sam Korsky.

2. Let $m$, $n$, and $x$ be positive integers. Prove that

   $$\sum_{i=1}^{n} \min\left(\left\lfloor \frac{x}{i} \right\rfloor, m\right) = \sum_{i=1}^{m} \min\left(\left\lfloor \frac{x}{i} \right\rfloor, n\right).$$

   *Proposed by Yang Liu.*

   **Solution 1.** Both sides count the number of entries of an $m \times n$ multiplication table that are at most $x$, as desired. ■

   This problem and solution were proposed by Yang Liu.

   **Solution 2.** We induct on $x$ for fixed $m$ and $n$. Note that it is trivial for $x = 0$ because both sides are 0. Now, say it is true for $x - 1$, and let's prove it is true for $x$. Note that the left increments for every value $i \leq n$ that has $\frac{x}{i} \leq m$ with $i$ dividing $x$. So it increments by 1 for every divisor of $x$ that is at least $\frac{x}{n}$ and at most $m$ (the $\frac{x}{i}$). The RHS increments by 1 for every divisor of $x$ that is at least $\frac{x}{m}$ and at most $n$ similarly. These are the same because $r$ dividing $x$ is in one category if and only if $\frac{x}{r}$ dividing $x$ is in the other. So we have a bijection, both increase by the same amount, and we are done by induction. ■

   This second solution was suggested by Ryan Alweiss.

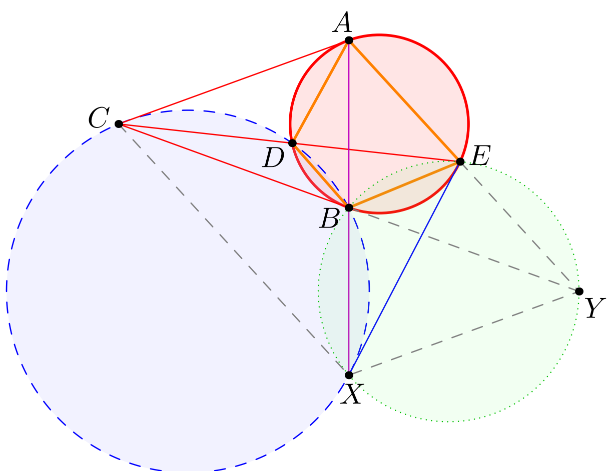
3. Let $\omega$ be a circle and $C$ a point outside it; distinct points $A$ and $B$ are selected on $\omega$ so that $\overline{CA}$ and $\overline{CB}$ are tangent to $\omega$. Let $X$ be the reflection of $A$ across the point $B$, and denote by $\gamma$ the circumcircle of triangle $BXC$. Suppose $\gamma$ and $\omega$ meet at $D \neq B$ and line $CD$ intersects $\omega$ at $E \neq D$. Prove that line $EX$ is tangent to the circle $\gamma$.

*Proposed by David Stoner.*

**Solution 1.** From $\angle CXB = \pi - \angle CDB = \angle EAB$, we find $AE \parallel CX$. Let $T \in \overline{CX}$ such that $AEXT$ is a parallelogram; then $\angle BTC = \pi - \angle AEB = \angle XBC$, and it follows that $\triangle BTC \sim \triangle XBC \Rightarrow (CX)(CT) = (CB)^2 = (CA)^2 \Rightarrow \triangle ATC \sim \triangle XAC$. Therefore $\angle CAT = \angle CXA = \angle CBT$, so $ACTB$ is cyclic. Finally, $\angle EXB = \angle BAT = \angle BCX$, and it follows that $\overline{EX}$ is tangent to $\omega$ as desired. ■

This problem and solution were proposed by David Stoner.

**Solution 2.**




Using directed angles, $\angle BXC = \angle BDC = \angle BDE = \angle BAE$ so $\overline{AE} \parallel \overline{CX}$. Construct parallelogram $AYXC$. As $\angle BEY = \angle BEA = \angle BAC = \angle BXY$, quadrilateral $BEXY$ is cyclic. Thus $\angle XCB = \angle BYE = \angle BXE$ as desired. ■

This second solution was suggested by Viswanath and mathdebam.

**Solution 3.** First note that $\angle ECX = \angle DBA = \angle CEA$ which implies that $EA \parallel CX$. Now let $F$ be the second intersection of line $AD$ with $\gamma$. We have that $\angle DFX = \angle ECX = \angle AEC = \angle DAC$ so $FC \parallel AX$. Therefore projecting points $C, D, B, X$ from $F$ onto line $AX$ yields that quadrilateral $CDBX$ is harmonic. Let $G = AB \cap ED$. Since line $AB$ is the polar of $C$ with respect to $\omega$ we have that $(C, G; D, E) = -1$ so by projecting $C, D, G, E$ from $X$ to circle $\gamma$ we have that $E$ must go to $X$ so $EX$ is tangent to $\omega'$ as desired. ■

This third solution was suggested by Sam Korsky.

**Solution 4.** Here is a solution with no auxiliary points at all. By angle chasing, $\triangle XAC \sim \triangle AEB$, whence

$$\frac{AX}{AE} = \frac{CX}{AB} = \frac{CX}{BC}.$$

Since $\angle BXC = \angle EAX$ also, we get $\triangle BXC \sim \triangle EAX$, thus $\angle BXE = \angle BCX$ as desired. ■

This fourth solution was suggested by linqaszayi.

**Remark.** An approach with complex numbers is also possible. Setting $\omega$ to be the unit circle, one can derive

$$d = \frac{b(2b+3a)}{2a+3b} \quad \text{and} \quad e = \frac{b(a+2b)}{2a+b}.$$

In fact, if one notices that $\overline{AE} \parallel \overline{CX}$ then the coordinates of $D$ can be bypassed, and point $E$ can be obtained directly.

It is even possible to approach the problem with Cartesian coordinates or by using barycentric coordinates on $\triangle ABC$.

4. Let $a > 1$ be a positive integer. Prove that for some nonnegative integer $n$, the number $2^{2^n} + a$ is not prime.

*Proposed by Jack Gurev.*

**Solution.** Let $m = v_2(a-1)$. Assume that $2^{2^m} + a = p$ is prime. It suffices to show there exists $n > m$ such that $2^{2^n} - 2^{2^m}$ is divisible by $p$.

Since

$$2^{2^n} - 2^{2^m} = 2^{2^m}\left(\left((2^{2^m}\right)^{2^{n-m}-1} - 1\right)$$

we can let

$$n = m + \phi\left(\frac{p-1}{2^m}\right)$$

which implies the conclusion. ■

This problem was proposed by Jack Gurev. This solution was given by Sam Korsky.

5. Let $m, n, k > 1$ be positive integers. For a set $S$ of positive integers, define $S(i,j)$ for $i < j$ to be the number of elements in $S$ strictly between $i$ and $j$. We say two sets $(X, Y)$ are a *fat* pair if

$$X(i,j) \equiv Y(i,j) \pmod{n}$$

for every $i, j \in X \cap Y$. (In particular, if $|X \cap Y| < 2$ then $(X, Y)$ is fat.)

If there are $m$ distinct sets of $k$ positive integers such that no two form a fat pair, show that $m < n^{k-1}$.

*Proposed by Allen Liu.*

**Solution.** Let the union of the sets be $T = \{a_1, a_2, \dots, a_\ell\}$ where the elements of $T$ are arranged in increasing order. For each element of $T$, color it randomly with one of $n$ colors $(1, 2, \dots, n)$. We say a set is good if its elements when arranged in increasing order have colors $a, a+1, \dots, a+k-1$ taken mod $n$ where $a$ can be any color. Now the fact that there is no fat pair means that only one good set can exist in each coloring. The probability that a good set exists is $\frac{1}{n^{k-1}}$ so we are done. (The

inequality is strict since we could end up coloring all elements of $T$ the same color.)
$\blacksquare$

This problem and solution were proposed by Allen Liu.