Two New Proofs of Leibniz's Inequality

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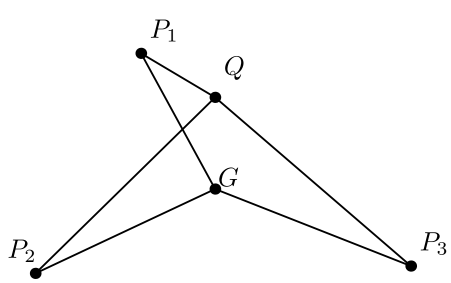
First Proof: Sums of Squares of Distances

1.1 A Quick Theorem

The first proof utilizes the following fact:

Theorem 1.1. *In the Cartesian plane, let $P_1, P_2, ...P_n$ be distinct points. For each point Q in the plane, let $X_Q = P_1Q^2 + P_2Q^2 + ... + P_nQ^2$. Then X_Q attains its minimum when Q is the centroid of the P_is, which we denote as G. (The centroid is just defined as the average of the coordinates of each point.)*



Proof. Indeed, we will show that the theorem reduces to nothing but the Trivial Inequality. First of all, let $P_i = (x_i, y_i)$ and let $G = (a, b)$ so that $a = \frac{x_1 + x_2 + ... + x_n}{n}, b = \frac{y_1 + y_2 + ... + y_n}{n}$

If $Q = (x, y)$ for arbitrary x, y then

$$X_Q = \sum_{i=1}^{n} (x_i - x)^2 + (y_i - y)^2$$

by the distance formula. Meanwhile, we clearly have

$$X_G = \sum_{i=1}^{n} (x_i - a)^2 + (y_i - b)^2$$

Thus showing $X_G \leq X_Q$ is equivalent to showing $\sum_{i=1}^{n}(x_i-x)^2+(y_i-y)^2-(x_i-a)^2-(y_i-b)^2 \geq 0$ or, by expansion, that

$$\sum_{i=1}^{n} x^2 + y^2 + 2ax_i + 2by_i - a^2 - b^2 - 2xx_i - 2yy_i \geq 0$$

However, we know that $\sum_{i=1}^{n} 2ax_i + 2by_i - 2xx_i - 2yy_i = 2n(a^2 + b^2 - ax - by)$ by definition of a, b so it suffices, after dividing by n, to show that $x^2 + y^2 - a^2 - b^2 + 2(a^2 + b^2 - ax - by) \geq 0$. But this is simply

$$(x-a)^2 + (y-b)^2 \geq 0$$

,meaning the proof is complete by the Trivial Inequality. □

1.2 The Proof of Leibniz's Inequality

Using the notations from earlier, we apply Theorem 1.1 on triangle ABC with $P_1 = A, P_2 = B, P_3 = C$, and $Q = O$, the circumcenter of ABC. Then if R is the circumradius of ABC we know that $X_G \leq X_O$ which is equivalent with $AG^2 + BG^2 + CG^2 \leq 3R^2$.

By Stewart's Theorem, the square of the length of the median from vertex A is $m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4}$. Then since $3AG = 2m_a$ our inequality is equivalent with

$$\sum_{cyc} \frac{2b^2 + 2c^2 - a^2}{9} \leq 3R^2$$

which reduces to exactly the inequality $9R^2 \geq a^2 + b^2 + c^2$ as desired.

1.3 Related Notes

Suppose H is the orthocenter of triangle ABC. Using the formula $AH = 2R\cos A$ along with the Law of Sines, the inequality $X_O \leq X_H$ reduces to Leibniz's Inequality. This gives us the chain $X_G \leq X_O \leq X_H$, and indeed, $X_G \leq X_H$ is exactly **Geolympiad Spring 2015 Shortlist G6** which can be found in [1].

Furthermore, Theorem 1.1 has a special result when $n = 3$: Namely that in a triangle ABC with centroid G and point P in the plane, we have

$$3X_P = 9GP^2 + AB^2 + BC^2 + CA^2$$

This formula immediately shows that X_P is minimized at $P = G$; then $PG = 0$.

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Second Proof: "Geometric Averaging"

2.1 Introduction to the "Geometric Averaging" Function

The second proof we present relies on ideas outlined in [2] where they are used to prove Euler's Inequality. Essentially, for A_0, B_0, C_0 angles of a triangle, we define

$$f(A_0, B_0, C_0) = (A_1, B_1, C_1) = \left(\frac{B+C}{2}, \frac{C+A}{2}, \frac{A+B}{2}\right)$$

Note that A_1, B_1, C_1 are still angles of a triangle- in fact, this function has a nice geometric interpretation. Take a triangle ABC with its circumcircle, draw the midpoints of arcs AB, BC, CA not containing the vertices of the triangle, and connect them to form a new triangle. This triangle has the angle measures described by our function, and the most important property is that **the circumradii of the old and new triangles are equal** since they have the same circumcircle.

Next, we present a simple fact:

Theorem 2.1. *As n goes to infinity, $f^n(A_0, B_0, C_0) = (A_n, B_n, C_n)$ becomes an equilateral triangle- that is, $A_n = B_n = C_n = 60$.*

Proof. This is very intuitive, but for a rigorous proof, note that $\max(A_i, B_i, C_i) - \min(A_i, B_i, C_i) = \frac{1}{2}[\max(A_{i-1}, B_{i-1}, C_{i-1}) - \min(A_{i-1}, B_{i-1}, C_{i-1})]$ for $i \geq 1$. (This is very easy to prove by without loss of generality assuming $A_{i-1} \geq B_{i-1} \geq C_{i-1}$.)

Now, this fact implies that as $n \to \infty$, we have $\max(A_n, B_n, C_n) - \min(A_n, B_n, C_n) = 0$, or that $A_n = B_n = C_n$, meaning that the triangle described is equilateral as desired. □

Now, let $g(\triangle XYZ) = XY^2 + YZ^2 + ZX^2$. Our next claim is that

2.2 The Key Inequality

Theorem 2.2. $g(\triangle A_iB_iC_i) \leq g(\triangle A_{i+1}B_{i+1}C_{i+1})$ *for* $i \geq 0$

This inequality essentially bridges the gap between $\triangle A_0B_0C_0$ and equilateral triangles by repeated iterations...

Proof. For convenience, let the angles of triangle $A_iB_iC_i$ be X, Y, Z. Then since $\triangle A_iB_iC_i$ and $\triangle A_{i+1}B_{i+1}C_{i+1}$ have a common circumradius which we denote as R, by Law of Sines it suffices to show that

$$4R^2(\sin^2 X + \sin^2 Y + \sin^2 Z) \leq 4R^2\left(\sin^2\left(\frac{X+Y}{2}\right) + \sin^2\left(\frac{Y+Z}{2}\right) + \sin^2\left(\frac{Z+X}{2}\right)\right)$$

We cancel $4R^2$ from both sides and subtract both sides of the inequality from 3 to get, using $\sin^2 n + \cos^2 n = 1$, that

$$\cos^2 X + \cos^2 Y + \cos^2 Z \geq \cos^2\left(\frac{X+Y}{2}\right) + \cos^2\left(\frac{Y+Z}{2}\right) + \cos^2\left(\frac{Z+X}{2}\right)$$

Next, we apply the identity which states that in any triangle ABC, $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$, a proof and much more of which can be found in [3], our inequality is equivalent with

$$1 - 2\cos X\cos Y\cos Z \geq 1 - 2\cos\left(\frac{X+Y}{2}\right)\cos\left(\frac{Y+Z}{2}\right)\cos\left(\frac{Z+X}{2}\right)$$

Finally, we subtract 1, divide by 2, and rearrange, and what we want to show is that

$$\cos\left(\frac{X+Y}{2}\right)\cos\left(\frac{Y+Z}{2}\right)\cos\left(\frac{Z+X}{2}\right) \geq \cos X\cos Y\cos Z$$

But now, we take the natural logarithm of both sides. Defining $h(x) = \ln\cos x$, our inequality becomes

$$h\left(\frac{X+Y}{2}\right) + h\left(\frac{Y+Z}{2}\right) + h\left(\frac{Z+X}{2}\right) \geq h(X) + h(Y) + h(Z)$$

We may quickly verify by calculus that the second derivative of $h(x)$ is $-\sec^2(x)$ which is always nonpositive, meaning h is a concave function. Thus, by Jensen, we have

$$h\left(\frac{X+Y}{2}\right) \geq \frac{h(X)+h(Y)}{2}$$

and summing the cyclic variants gives the desired result. $\square$

2.3 The Proof of Leibniz's Inequality

After the previous claim, we are almost done. We originally wanted to show that $g(\triangle A_0B_0C_0) \leq 9R^2$. But by Theorem 2.2, we have

$$g(\triangle A_0B_0C_0) \leq g(\triangle A_1B_1C_1) \leq g(\triangle A_2B_2C_2) \leq ... \leq g(\triangle A_nB_nC_n)$$

as n approaches infinity.Thus, if we show $g(\triangle A_nB_nC_n) \leq 9R^2$, we will be done.

But by Theorem 2.1, we see that $g(\triangle A_nB_nC_n) = 9R^2$! This is because $A_nB_nC_n$ becomes equilateral, meaning each of its side lengths are $\sqrt{3}$ times the circumradius of the triangle, which gives us the desired!

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Concluding Remarks

We have now shown two completely new proofs of Leibniz's Inequality and hopefully they reveal why the inequality holds- in particular, the formula for OH seems rather unmotivated and random, so we hope our proofs give more order to this inequality. Indeed, the method used in the second proof is quite powerful and sheds light on a number of inequalities.

3.1 References

[1]: http://artofproblemsolving.com/community/c7419h1077524p4716586

[2]: http://www.artofproblemsolving.com/community/c4h574490p3383299

[3]: Simple Trigonometric Substitutions with Broad Results by Vardan Verdiyan, Daniel Campos Salas