The Erdos-Mordell inequality

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Let ABC be a triangle and an interior point M. We denote by $AM = x_1, \ BM = x_2, \ CM = x_3$. The distances of the point M from BC, CA, AB are denoted by $p_1, \ p_2, \ p_3$.
Erdos-Mordell inequality asserts.

$$x_1 + x_2 + x_3 \geq p_1 + p_2 + p_3$$

The above problem proposed by P.Erdos in the American Mathematical Monthly in 1935 and solved byI.J.Mordell and D.F.Borrow in 1937. Later many Mathematicians obtained solutions and for a long time the problem was in the air. Some classical solutions can be found in the following main sources:

1.Geometric Inequalities by O.Bottema, R.Z.Djordjevic, R.R.Janic,D,S.Mitrinovic, P.M.Vasic,Wolters-Noordhoff Pub.
2.Recent Advances in Geometric Inequalities by D.S.Mitrinovic, J.E.Pecaric and V.Volonec. Klwver Academic pub.
3.Geometric Inequalities by N.D.Kazarinoff. Random House.
4.Plane Geometry and its Groups by H.W.Guggenaimer. Holden Day.
5.Introduction to Geometry by H.S.M.Coxeter. John Wiley and sons.
Many years ago I found some new (at least for me) proofs and the most interesting are sortly exposed below. Some of them, I am sure, have been obtained and by others Matimaticians.Anywhere, I believe that this article is useful especialy for young Mathematicians.

Proof 1

From the point M we drop the perpenticulars MD, ME, MF to BC, CA, AB respectively and then the perpenticulars EE', FF' to BC We easily see that: $\angle MEE' = C \ \ and \ \ \angle MFF' = B$. We have.

$$FE = AM.sinA = x_1 sinA$$

Obviously
$FE \geq F'E' = F'D + DE' = p_3 sinMFF' + p_2 sinMEE' = p_2 sinB + P_3 sinC$
That is:

$$x_1 \geq p_3 \frac{sinB}{sinA} + p_2 \frac{sinC}{sinA}$$

Two similar inequalities follow and adding we take.

$$\sum x_i \geq \sum p_1 \left(\frac{sinB}{sinC} + \frac{sinC}{sinB}\right) \geq 2(p_! + p_2 + p_3)$$

see fig.1

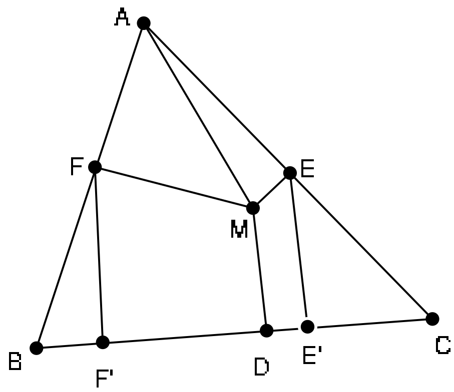

Figure 1:

Proof 2

We denote by D,E,F the feets of the perpendiculars of an interior point M to the sides of the triangle ABC. Using the same notation as in the proof 1, we have.

To the triangle DEF.

$$p_1.EF \geq 2[(DMF) + (DME)] = p_1 p_3 sinB + p_1 p_2 sinC$$

Therefore

$$EF \geq p_3 sinB + p_2 sinC \quad or \quad x_1 sinA \geq p_3 sinB + p_2 sinC$$

or,

$$x_1 \geq p_3 \frac{sinB}{sinA} + p_2 \frac{sinC}{sinA} \quad etc.$$

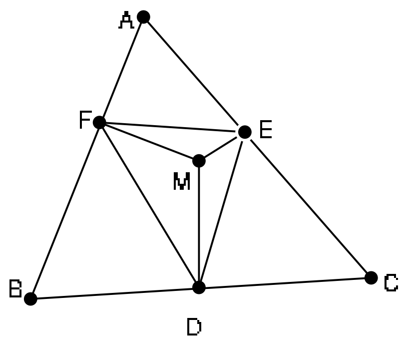

Figure 2:

Proof 3

Let Ax the symmetric of the AM relative the bissectrice of the angle A and BB', CC' the distances of B andC from Ax.

We have:

$a \geq BB' + CC' = c.sinMAC + b.sinMAB = c.\frac{p_2}{x_1} + b.\frac{p_3}{x_1}$

Therefore

$$a.x_1 \geq c.p_@ + b.p_3 \quad and \quad x_1 \geq p_2 \frac{c}{a} + p_3 \frac{b}{a} \quad etc.$$

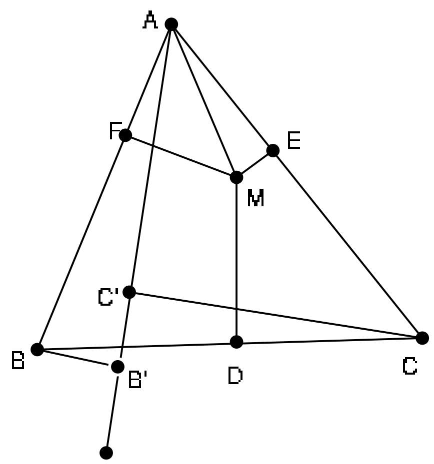

Figure 3:

Proof 4

We drop the perpenticulars BB', AA', CC' to the line EF. Obviously we have:

$$B'C' = c.cosAFA' + b.cosAEA' = c.sinMFA' + b.sinMEA' \leq a \qquad (1)$$

or

$$\frac{sinMFA'}{p_2} = \frac{sinMEA'}{p_3} = \frac{sinA}{EF} = \frac{1}{x_1} \qquad (2)$$

From (1) and (2) follows:

$$c.\frac{p_2}{x_1} + b.\frac{p_3}{x_1} \leq a$$

or

$$p_3\frac{b}{a} + p_2\frac{c}{a} \leq x_1 \quad etc.$$

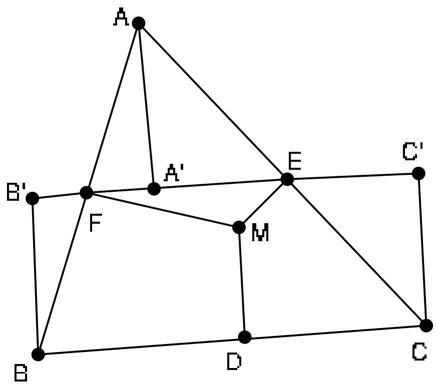

Figure 4:

Proof 5

Let D,E,F the feets of the perpendiculars on the sides BC, CA, AB respectively. We drop the perpendiculars EE', FF' to BC.We will have:

$$x_1 + x_2 + x_3 \geq \sum \frac{E'F'}{EF} x_1$$

But

$$E'F' = p_3 sinB + p_2 sinC \qquad and \quad EF = x_1 sinA$$

Therefore

$$\sum x_i \geq \sum (p_3 \frac{sinB}{sinA} + p_2 \frac{sinC}{sinA})$$

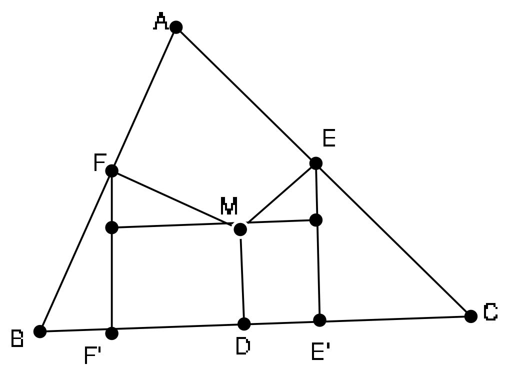

Figure 5:

Proof 6

The feets of the perpendiculars from the point M to the sides BC, CA, AB are the points D,E,F respectively. From E,F we drop the perpendiculars EE' and FF' to DM. We obviously have:

$$EF \geq EE' + FF' = p_3 sinB + p_2 sinC$$

That is

$$x_1 sinA \geq p_3 sinB + p_2 sinC$$

Cyclicaly we take two other inequalities etc.

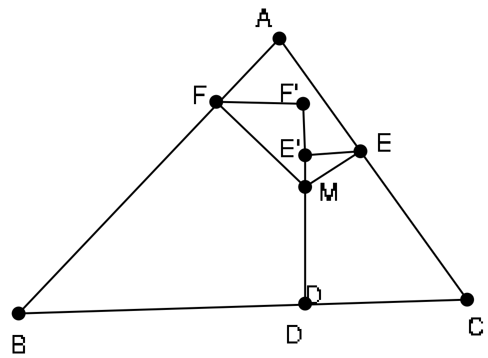

Figure 6:

Proof 7

The antiparallel from the point M intersects AB to the oint B' and the side AC to the point C'. We have $AB'.AB = AC'.AC$

We also easily see:

$$x_1.B'C' \geq p_2.AC' + p_3.AB'$$

or

$$x_1 \geq p_2.\frac{AC'}{B'C'} + p_3.\frac{AB'}{B'C'} = p_3\frac{sinC}{sinA} + p_3.\frac{sinB}{sinA} \quad etc.$$

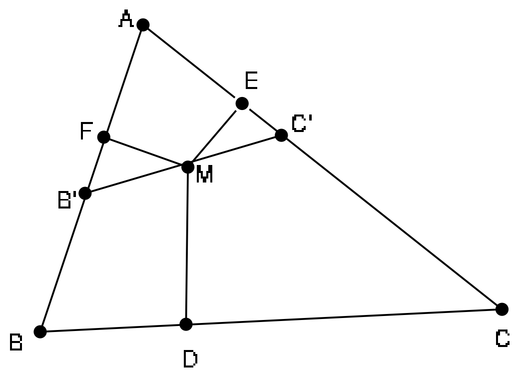

Figure 7:

Proof 8

The circle ABC intersects the line AM to the point A'. The Ptolemy's theorem to the inscribed ABA'C is:

$$AA'.BC = A'C.AB + A'B.AC$$

Let A'D and A'E the distances of the point A' from AC and AB respectively. Obviously $A'C \geq A'D \ \ and \ \ A'B \geq A'E$

Therefore

$$AA'.a \geq A'D.c + A'E.b$$

or

$$1 \geq \frac{A'D.c}{AA'.a} + \frac{A'E.b}{AA'.a}$$

but,

$$\frac{A'D}{AA'} = \frac{p_2}{x_1}, \quad and \quad \frac{A'E}{AA'} = \frac{p_3}{x_1}$$

Hance,

$$x_1 \geq p_2\frac{c}{a} + p_3\frac{b}{a}$$

etc.

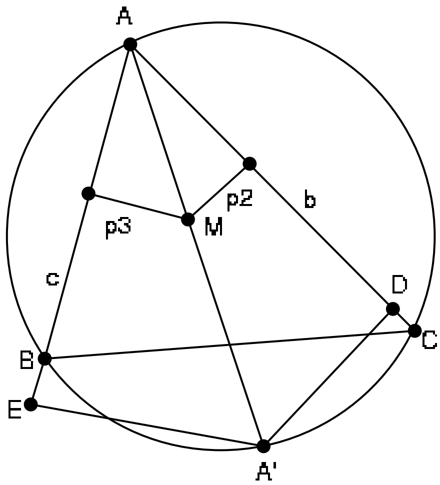

Figure 8:

Proof 9

The line DM inersects the circle AFE at the point A'. The triangle A'FE is similar to the triangle ACB. Let d_e, d_f, the distances of the points E and F from A'M.

We obviously have: $A'E.ME = 2Rd_e, \quad and \quad A'F.FM = 2Rd_F$, where R is the radius of the circle AEF.

Adding we take

$$p_2.A'E + p_3.A'F \geq x_1.FE \quad or \quad x_1 \geq p_2.\frac{A'E}{FE} + \frac{A'F}{FE}$$

or

$$x_1 \geq p_2.\frac{sinC}{sinA} + p_3.\frac{sinB}{sinA}$$

etc.

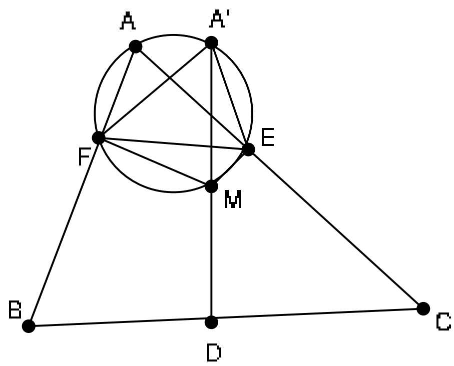

Figure 9:

Proof 10

The circle BMC intersects the line AM to the point A' and the sides AB and AC to the points B' and C' respectively. Wewill have:

$$x_1.B'C' \geq p_3.AB' + p_2.AC'$$

The triangles AB'C' and ACB are similar. Therefore

$$x_1 \geq p_3.\frac{AB'}{B'C'} + p.\frac{AC'}{B'C'} = p_3\frac{b}{a} + p_2\frac{c}{a}$$

etc.

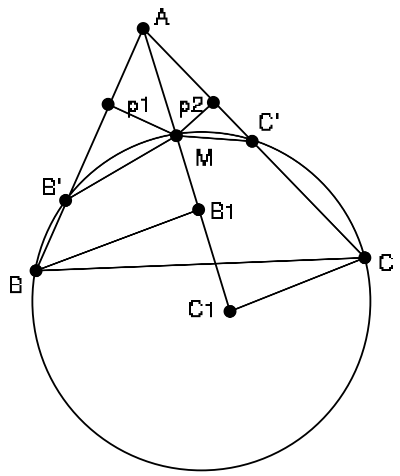

Figure 10:

Proof 11

The circle BMC intersects AM in A' AB in B' and AC in C'. The triagles AMB and ABA' are similar as well the triangles AMC and ACA'.Therefore from

$$\frac{p_3}{BB_1} = \frac{AM}{AB} \quad follows \quad p_3.AB = AM.B_1$$

and

$$\frac{p_2}{CC_1} = \frac{AM}{AC} \quad follows \quad p_2.AC = AM.CC_1$$

Hence

$$p_3 c + p_2 b = AM(BB_! + CC_1) \leq AM.a$$

and finaly

$$p_3\frac{c}{a} + p_2\frac{b}{a} \leq x_1. \ \ etc.$$

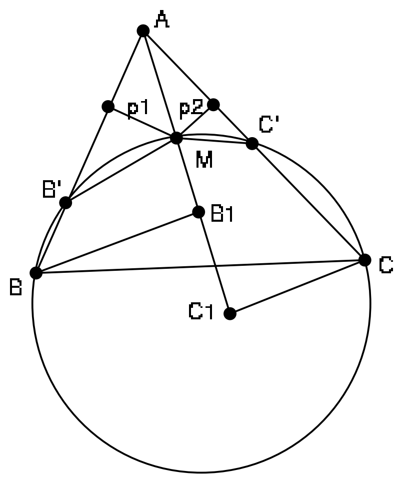

Figure 11:

Proof 12

Let P be a point on the side BC so that $\angle BAD = \angle PAC$.We have $AP.BC \geq PE'.b + PF'.C$ where by E',F' are the feets of the perpendiculars from P to AB. AC respectively.

We easily see that:

$$1 \geq \frac{PE'}{AP}.\frac{b}{a} + \frac{PF'}{AP}.\frac{c}{a}$$

but,

$$\frac{PE'}{AP} = \frac{p_3}{x_1}, \quad \frac{PF'}{AP} = \frac{p_2}{x_1}$$

and finaly

$$1 \geq \frac{p_3 b}{x_1 a} + \frac{p_2 c}{x_1 a}$$

etc.

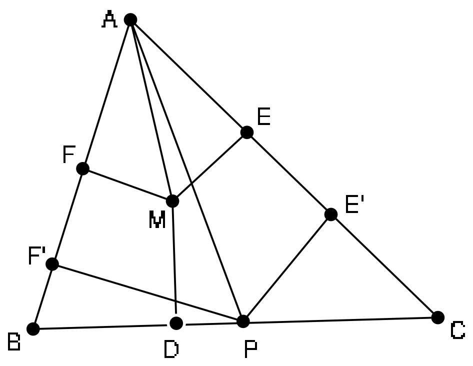

Figure 12:

Proof 13

We consider the circle AFME. The parallel line from m to EF intersects the circle to the point M'. Then we drop the perpenticulars M'E', M'F' to AC,AB respectively. We denote $M'E' = p_2', \quad M'F' = p_3'$. We also see that arc EM=arc M'F. It follows:

$$\frac{p_3'}{AM'} = \frac{p_2}{AM}, \qquad \frac{p_2'}{AM'} = \frac{p_3}{AM}$$

We obviously have:

$$AM'.a \geq p_2'.c + p_3'.b$$

or

$$1 \geq \frac{p_2'c}{AM'a} + \frac{p_3'b}{AM'} = \frac{p_2 c}{x_1 a} + \frac{p_3 b}{x_1 a}$$

etc.

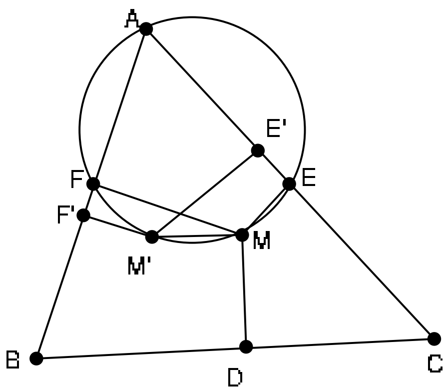

Figure 13: