Ex-Lincoln Math Olympiad

Year: 2015

Day: 1

17th ELMO
Pittsburgh, PA

June 20, 2015
1:00 PM – 6:00 PM

Problem 1. Define the sequence $a_1 = 2$ and $a_n = 2^{a_{n-1}} + 2$ for all integers $n \geq 2$. Prove that a_{n-1} divides a_n for all integers $n \geq 2$.

Problem 2. Let m, n, and x be positive integers. Prove that

$$\sum_{i=1}^{n} \min\left(\left\lfloor \frac{x}{i} \right\rfloor, m\right) = \sum_{i=1}^{m} \min\left(\left\lfloor \frac{x}{i} \right\rfloor, n\right).$$

Problem 3. Let ω be a circle and C a point outside it; distinct points A and B are selected on ω so that $\overline{CA}$ and $\overline{CB}$ are tangent to ω. Let X be the reflection of A across the point B, and denote by γ the circumcircle of triangle BXC. Suppose γ and ω meet at $D \neq B$ and line CD intersects ω at $E \neq D$. Prove that line EX is tangent to the circle γ.

Problem 4. Let $a > 1$ be a positive integer. Prove that for some nonnegative integer n, the number $2^{2^n} + a$ is not prime.

Problem 5. Let $m, n, k > 1$ be positive integers. For a set S of positive integers, define $S(i, j)$ for $i < j$ to be the number of elements in S strictly between i and j. We say two sets (X, Y) are a *fat* pair if

$$X(i, j) \equiv Y(i, j) \pmod{n}$$

for every $i, j \in X \cap Y$. (In particular, if $|X \cap Y| < 2$ then (X, Y) is fat.)

If there are m distinct sets of k positive integers such that no two form a fat pair, show that $m < n^{k-1}$.

Time limit: 5 hours.
Each problem is worth 7 points.