The uvw method

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1 Basic Concepts

The basic concept of the method is this:

With an inequality in the numbers $a, b, c \in \mathbb{R}$, we write everything in terms of $3u = a+b+c, 3v^2 = ab+bc+ca, w^3 = abc$. The condition $a, b, c \geq 0$ is given in many inequalities. In this case, we see that $u, v^2, w^3 \geq 0$. But if this is not the case, u, v^2, w^3 might be negative. E.g. $a = -1, b = -2, c = 3$ we get $3v^2 = -7$. So don't be confused! Sometimes v^2 can be negative!

The Idiot-Theorem: $u \geq v \geq w$, when $a, b, c \geq 0$.

Proof: $9u^2 - 9v^2 = a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}\sum_{cyc}(a-b)^2 \geq 0$. And hence $u \geq v$. From the AM-GM inequality we see that $v^2 = \frac{ab+bc+ca}{3} \geq \sqrt[3]{a^2b^2c^2} = w^2$, and hence $v \geq w$, which concludes $u \geq v \geq w$. $\square$

The idiot-theorem is usually of little use alone, but it is important to know.

Sometimes we introduce the ratio $t = u^2 : v^2$. If $u > 0, v = 0$ we define $t = \infty$, and if $u < 0, v = 0$ we define $t = -\infty$. If $u = v = 0$ we set $t = 1$. When $a, b, c \geq 0$, we see that $t \geq 1$. And we always see that $|t| \geq 1$.

The UVW-Theorem: *Given $u, v^2, w^3 \in \mathbb{R}$: $\exists a, b, c \in \mathbb{R}$ such that $3u = a + b + c$, $3v^2 = ab + bc + ca$, $w^3 = abc$ if and only if:*

$u^2 \geq v^2$ *and* $w^3 \in \left[3uv^2 - 2u^3 - 2\sqrt{(u^2-v^2)^3}, 3uv^2 - 2u^3 + 2\sqrt{(u^2-v^2)^3}\right]$

Proof: Define $f(t) = t^3 - 3ut^2 + 3v^2t - w^3$. Let a, b, c be its roots. By vietés formulas we see that $3u = a + b + c, 3v^2 = ab + bc + ca, w^3 = abc$.

Lemma 1: $a, b, c \in \mathbb{R} \iff (a-b)(b-c)(c-a) \in \mathbb{R}$.

Proof of lemma 1: It's trivial to see that $a, b, c \in \mathbb{R} \Rightarrow (a-b)(b-c)(c-a) \in \mathbb{R}$. Then I'll show $a, b, c \notin \mathbb{R} \Rightarrow (a-b)(b-c)(c-a) \notin \mathbb{R}$. Let $z \notin \mathbb{R}$ be a complex number such that $f(z) = 0$. Then $f(\overline{z}) = \overline{f(z)} = \overline{0} = 0$, so if z is a complex root in $f(t)$, so is $\overline{z}$. Assume wlog because of symmetry that $a = z, b = \overline{z}$. Then $(a-b)(b-c)(c-a) = -(z-\overline{z})|z-c|^2 \notin \mathbb{R}$. The last follows from $(a-b)(b-c)(c-a) \neq 0$, $i(z-\overline{z}) \in \mathbb{R}$, and $|z-c|^2 \in \mathbb{R}$. $\square$

It's obvious that $x \in \mathbb{R} \iff x^2 \in [0; +\infty)$. So $\exists a, b, c \in \mathbb{R}$ such that $3u = a+b+c, 3v^3 = ab + bc + ca, w^3 = abc$ if and only if $(a-b)^2(b-c)^2(c-a)^2 \geq 0$, where a, b, c are the roots of $f(t)$.

But $(a-b)^2(b-c)^2(c-a)^2 = 27(-(w^3 - (3uv^2 - 2u^3))^2 + 4(u^2-v^2)^3) \geq 0 \iff 4(u^2-v^2)^3 \geq (w^3 - (3uv^2 - 2u^3))^2$. From this we would require $u^2 \geq v^2$. But $4(u^2-v^2)^3 \geq (w^3 - (3uv^2 - 2u^3))^2 \iff$

$2\sqrt{(u^2-v^2)^3} \geq |w^3 - (3uv^2 - 2u^3)|$

When $w^3 \geq (3uv^2 - 2u^3)$ it's equivalent to:

$w^3 \leq 3uv^2 - 2u^3 + 2\sqrt{(u^2-v^2)^3}$

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So $(a-b)^2(b-c)^2(c-a)^2 \geq 0 \iff w^3 \in [3uv^2 - 2u^3; 3uv^2 - 2u^3 + 2\sqrt{(u^2-v^2)^3}] \cap$

$[3uv^2 - 2u^3 - 2\sqrt{(u^2-v^2)^3}; 3uv^2 - 2u^3] \iff w^3 \in [3uv^2 - 2u^3 - 2\sqrt{(u^2-v^2)^3}; 3uv^2 - 2u^3 + 2\sqrt{(u^2-v^2)^3}]$ and $u^2 \geq v^2$. □

The Positivity-Theorem: $a, b, c \geq 0 \iff u, v^2, w^3 \geq 0$

Proof: The case $a, b, c \geq 0 \Rightarrow u, v^2, w^3 \geq 0$ is obvious. Now the proof of $a < 0$ or $b < 0$ or $c < 0 \Rightarrow u < 0$ or $v^2 < 0$ or $w^3 < 0$. If there is an odd number of negative numbers in (a, b, c) then w^3 is negative. Else there is two negative. Assume wlog because of symmetry $a \geq 0, b, c < 0$. Let $b = -x, c = -y$. So $3u = a - x - y$, $3v^2 = xy - a(x+y)$, and $x, y > 0$. $u \geq 0 \iff a \geq x + y$. If $a \geq x + y$ then $xy - a(x+y) \leq -x^2 - xy - y^2$, and hence not both u and v^2 can be positive, and we are done.

Theorem: All symetric polynomials in a, b, c can be written in terms of u, v^2, w^3. (No one dividing. E.g. the term $\frac{u}{w^3}$ is not allowed)

Proof: Well-known.. Use induction, and the fact that $a^n + b^n + c^n$ always can be repressented in terms of u, v^2, w^3. (Will post full proof later)

2 Qualitative estimations

It can be really tedious to write everything in terms of u, v^2, w^3. Because of this it can be really useful to know some qualitative estimations.

We already have some bounds on w^3, but they are not always (that is, almost never) "nice". The squareroot tend to complicate things, so there is needed a better way, than to use the bounds always!

If you haven't noticed: Many inequalities have equality when $a = b = c$. Some have when $a = b, c = 0$, and some again for $a = b = kc$ for some k. There are rarely equality for instance when $a = 3, b = 2, c = 1$ - although it happens.

There is a perfectly good reason for this: When we fix two of u, v^2, w^3, then the third assumes it's maximum if and only if two of a, b, c are equal!

This is really the nicest part of this paper. (I have never seen this idea applied before!)

It is not important to memorize the full proof. Both II, III will be very close to I (actually i copy-pasted most of it, just to save time). Just remember the main ideas!

"Tejs's Theorem":p *Assume that we are given the constraint $a, b, c \geq 0$. Then the following is true:*

(I) When we have fixed u, v^2 and there exists at least one value of w^3 such that there exists $a, b, c \geq 0$ corresponding to u, v^2, w^3: Then w^3 has a global maximum and minimum. w^3 assumes maximum only when two of a, b, c are equal, and minimum either when two of a, b, c are equal or when one of them are zero.

(II) When we have fixed u, w^3 and there exists at least one value of v^2 such that there exists $a, b, c \geq 0$ corresponding to u, v^2, w^3: Then v^2 has a global maximum and minimum. v^2 assumes maximum only when two of a, b, c are equal, and minimum only when two of a, b, c are equal.

(III) When we have fixed v^2, w^3 and there exists at least one value of u such that there exists $a, b, c \geq 0$ corresponding to u, v^2, w^3: Then u has a global maximum and minimum. u assumes maximum only when two of a, b, c are equal, and minimum only when two of a, b, c are equal.

Proof:

(I) We'll use something from the earlier proofs(The lemma from the UVW-theorem plus the positivity theorem): There exists $a, b, c \geq 0$ corresponding to u, v^2, w^3 if and only if $-(w^3 - 3uv^2 + 2u^3)^2 + 4(u^2 - v^2)^3, u, v^2, w^3 \geq 0$ (Hence $u, v^2 \geq 0$, and we will implicitly assume $w^3 \geq 0$ in the following.). Setting $x = w^3$ it is equivalent to $p(x) = Ax^2 + Bx + C \geq 0$ for some $A, B, C \in \mathbb{R}$. (Remember we have fixed u, v^2) Since $A = -1$ we see $A < 0$. Since there exists $x \in \mathbb{R}$ such that $p(x) \geq 0$ and obviously $p(x) < 0$ for sufficiently large x, we conclude that it must have at least one positive root. Let α be the largest positive root, then it is obvious that $p(x) < p(\alpha) = 0 \; \forall x > \alpha$. That is: The biggest value x can attain is α, so $x = w^3 \leq \alpha$. This can be attained by the previous proof, and it does so only when two of a, b, c are equal (because $(a - b)(b - c)(c - a) = 0$). Let β be the smallest positive root of $p(x)$. If $p(x) < 0$ when $x \in [0; \beta)$ then x is at least β and $x = w^3 \geq \beta$, with equality only if two of a, b, c are equal, since $(a - b)(b - c)(c - a) = 0$. If $p(x) \geq 0$ when $x \in [0; \beta)$ then $x = w^3 \geq 0$ with equality when one of a, b, c are zero, since $w^3 = abc = 0$. □

(II) We'll use something from the earlier proofs(The lemma from the UVW-theorem plus the positivity theorem): There exists $a, b, c \geq 0$ corresponding to u, v^2, w^3 if and only if $-(w^3 - 3uv^2 + 2u^3)^2 + 4(u^2 - v^2)^3, u, v^2, w^3 \geq 0$ (Hence $u, w^3 \geq 0$, and we will implicitly assume $v^2 \geq 0$ in the following.). Setting $x = v^2$ it is equivalent to $p(x) = Ax^3 + Bx^2 + Cx + D \geq 0$ for some $A, B, C \in \mathbb{R}$. (Remember we have fixed u, w^3) Since $A = -4$ we see $A < 0$. Since there exists $x \in \mathbb{R}$ such that $p(x) \geq 0$ and obviously $p(x) < 0$ for sufficiently large x, we conclude that it must have at least one positive root. Let α be the largest positive root, then it is obvious that $p(x) < p(\alpha) = 0 \; \forall x > \alpha$. That is: The biggest value x can attain is α, so $x = v^2 \leq \alpha$. This can be attained by the previous proof, and it does so only when two of a, b, c are equal (because $(a - b)(b - c)(c - a) = 0$). Let β be the smallest positive root of $p(x)$. If $p(x) < 0$ when $x \in [0; \beta)$ then x is at least β and $x = v^2 \geq \beta$, with equality only if two of a, b, c are equal, since $(a - b)(b - c)(c - a) = 0$. If $p(x) \geq 0$ when $x \in [0; \beta)$ then $x = v^2 \geq 0$ with equality when two of a, b, c are zero, since $3v^2 = ab + bc + ca = 0$, and then two of them have to be equal. □

(II) We'll use something from the earlier proofs(The lemma from the UVW-theorem plus the positivity theorem): There exists $a, b, c \geq 0$ corresponding to u, v^2, w^3 if and only if $-(w^3 - 3uv^2 + 2u^3)^2 + 4(u^2 - v^2)^3, u, v^2, w^3 \geq 0$ (Hence $v^2, w^3 \geq 0$, and we will implicitly assume $u \geq 0$ in the following.). Setting $x = u$ it is equivalent to $p(x) = Ax^3 + Bx^2 + Cx + D \geq 0$ for some $A, B, C \in \mathbb{R}$. (Remember we have fixed v^2, w^3) Since $A = -4w^3$ we see $A < 0$. Since there exists $x \in \mathbb{R}$ such that $p(x) \geq 0$ and obviously $p(x) < 0$ for sufficiently large x, we conclude that it must have at least one positive root. Let α be the largest positive root, then it is obvious that $p(x) < p(\alpha) = 0 \; \forall x > \alpha$. That is: The biggest value x can attain is

α, so $x = u \le \alpha$. This can be attained by the previous proof, and it does so only when two of a, b, c are equal (because $(a-b)(b-c)(c-a) = 0$). Let β be the smallest positive root of $p(x)$. If $p(x) < 0$ when $x \in [0; \beta)$ then x is at least β and $x = u \ge \beta$, with equality only if two of a, b, c are equal, since $(a-b)(b-c)(c-a) = 0$. If $p(x) \ge 0$ when $x \in [0; \beta)$ then $x = u \ge 0$ with equality when all three of a, b, c are zero, since $3u = a + b + c = 0$, and then two of them have to be equal. □

Corrolar: Every symmetric inequality of degree ≤ 5 has only to be proved when $a = b$ and $a = 0$.

Proof: Since it can be written as the symmetric functions it is linear in w^3. Hence it is either increasing or decreasing. So we only have to check it when two of a, b, c are equal or when one of them are zero. Because of symmetry we can wlog assume $a = b$ or $c = 0$. □

3 Shortcuts

I will write some well known identities. It would be very tedious if you had to argument that $a^2 + b^2 + c^2 = 9u^2 - 6v^2$ every time you used it, wouldn't it?

The "must remember":

$(a-b)^2(b-c)^2(c-a)^2 = 27(-(w^3 - 3uv^2 + 2u^3)^2 + 4(u^2 - v^2)^3)$

Schur's ineq for third degree:

$\sum_{cyc} a(a-b)(a-c) \ge 0 \iff w^3 + 3u^3 \ge 4uv^2$.

Some random identities:

$(a-b)(b-c)(c-a) = \sum_{cyc} b^2a - ab^2$

$a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 9u^2 - 6v^2$

$(ab)^2 + (bc)^2 + (ca)^2 = (ab+bc+ca)^2 - 2abc(a+b+c) = 9v^4 - 6uw^3$

$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \iff a^3+b^3+c^3 = 27u^3-27uv^2+3w^3$

$a^4 + b^4 + c^4 = (a^2+b^2+c^2)^2 - 2((ab)^2 + (bc)^2 + (ca)^2) = (9u^2 - 6v^2)^2 - 2(9v^2 - 6uw^3) = 81u^4 - 108u^2v^2 + 18v^4 + 12uw^3$

4 Applications

The uvw-method is (unlike for instance complex numbers in geometry :p) not always the best choice. When dealing with squareroots, inequalities of a very high degree og simply non-symmetric or more than 4 variable inequalities, you probably want to consider using another technique. (But e.g. there is a solution to one with a square root here, so it is possible.)

I will show some examples of the use of the uvw-method. Most of the problems will also have solutions without using the uvw-method, but I will not provide those.

IMO prob. 3, 06: Determine the least real number M such that the inequality

$$\left|ab\left(a^2-b^2\right)+bc\left(b^2-c^2\right)+ca\left(c^2-a^2\right)\right| \le M\left(a^2+b^2+c^2\right)^2$$

holds for all real numbers a, b and c

Solution:

Notice the identity $ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2) = (a-b)(b-c)(a-c)(a+b+c)$. So we should prove (after squaring):

$(a-b)^2(b-c)^2(c-a)^2(a+b+c)^2 \leq M^2(a^2+b^2+c^2)^4$

Introduce the notations: $3u = a+b+c, 3v^2 = ab+bc+ca, w^3 = abc, u^2 = tv^2$. (Thus $t^2 \geq 1$) (Notice that it is possible that $v^2 \leq 0$, so we don't necesarily have $v \in \mathbb{R}$, but always $v^2 \in \mathbb{R}$)

Obviously $a^2+b^2+c^2 = 9u^2-6v^2$ and $(a-b)^2(b-c)^2(c-a)^2 = 3^3(-(w^3-(3uv^2-2u^3))^2+4(u^2-v^2)^3)$

So we just have to prove:

$3^5 \cdot u^2 \cdot (-(w^3-(3uv^2-2u^3))^2+4(u^2-v^2)^3) \leq M^2(9u^2-6v^2)^4 \iff$

$3u^2(-(w^3-(3uv^2-2u^3))^2+4(u^2-v^2)^3) \leq M^2(3u^2-2v^2)^4$.

Now $-(w^3-(3uv^2-2u^3))^2 \leq 0$. So it suffices to prove that:

$12u^2(u^2-v^2)^3 \leq M^2(3u^2-2v^2)^4$

Divide through with v^8:

$12t(t-1)^3 \leq M^2(3t-2)^4$

If t is positive, then $t \geq 1$:

$M^2 \geq \frac{12t(t-1)^3}{(3t-2)^4}$

Consider the function $f(t) = \frac{12t(t-1)^3}{(3t-2)^4}$.

Then $f'(t) = \frac{12(t-1)^2(t+2)}{(3t-2)^5} > 0 \forall t \in [1; +\infty)$.

So we just need to consider the expression $\frac{12t(t-1)^3}{(3t-2)^4}$ as $t \to +\infty$.

Writing it as $\frac{12t(t-1)^3}{(3t-2)^4} = \frac{12(1-\frac{1}{t})^3}{(3-\frac{2}{t})^4}$. It's obvious that $\frac{12t(t-1)^3}{(3t-2)^4} \to \frac{4}{27}$ when $t \to +\infty$. So if t is positive then $M \geq \frac{2}{3\sqrt{3}}$

Now assume t is negative. Then $t \leq -1$:

$M^2 \geq \frac{12t(t-1)^3}{(3t-2)^4}$

Consider the function $f(t) = \frac{12t(t-1)^3}{(3t-2)^4}$.

Then $f'(t) = \frac{12(t-1)^2(t+2)}{(3t-2)^5}$.

So we have a maximum when $t = -2$:

$f(-2) = \frac{3^4}{2^9}$

So $M^2 \geq \frac{3^4}{2^9} \iff M \geq \frac{9}{16\sqrt{2}}$.

Since $\frac{9}{16\sqrt{2}} \geq \frac{2}{3\sqrt{3}}$ we get that $M = \frac{9}{16\sqrt{2}}$ fullfilles the condition. To prove that M is a minimum, we have to find numbers a, b, c such that equality is archeived.

Looking at when we have minimum. ($w^3 = 3uv^2 - 2u^3$ and $v^2 = \frac{u^2}{-2}$) We get that roots in the polynomial $x^3-6x^2-6x+28$ will satisfy the minimum. But $x^3-6x^2-6x+28 = (x-2)(x^2-4x+14)$, so we can easily find a, b, c. Indeed when $(a, b, c) = (2, 2+3\sqrt{2}, 2-3\sqrt{2})$ there is equality and $M = \frac{9}{16\sqrt{2}}$ is the minimum!

Note: Sometimes it can be very time consuming to find the roots of a third degree polynomial. So if you just prove that $u^2 \geq v^2$ and $w^3 \in [3uv^2 - 2u^3 - 2\sqrt{(u^2-v^2)^3}; 3uv^2 - 2u^3 + 2\sqrt{(u^2-v^2)^3}]$, then you can say that the polynomial has three real roots for sure, even though you cannot tell exactly what they are. (It is very easy to do so in this example since $w^3 = 3uv^2 - 2u^3$) $\square$

From V. Q. B. Can: Let a, b, c be nonnegative real numbers such that $ab+bc+ca = 1$. Prove that

$$\frac{1}{2a+2bc+1} + \frac{1}{2b+2ca+1} + \frac{1}{2c+2ab+1} \geq 1$$

Solution:

This is a solution, which shows the full power of the uvw method. There is almost no calculation involved, and it is very easy to get the idea!

It's obvious equivalent to: $\sum_{cyc}(2a+2bc+1)(2b+2ca+1) \geq \prod_{cyc}(2a+2bc+1)$

Now fix u, v^2. Since LHS is of degree four and RHS is of degree six, there must exist A, B, C, D, E such that $LHS = Aw^3 + B$ and $RHS = Cw^6 + Dw^3 + E$. And we easily see $C = 8 > 0$. So we have to prove $g(w^3) = -Cw^6 + (D-A)w^3 + (B-E) \geq 0$, which is obviously concave in w^3. We only have to check the case where $g(w^3)$ are minimal. That is: Either when w^3 are minimal or maximal. So we only have to check the case where two of a, b, c are equal, or the case where one of a, b, c are zero.

It is left to the reader to prove these trivial cases. $\square$

Unknown origin: Let $a, b, c \geq 0$, such that $a+b+c = 3$. Prove that:

$$(a^2b + b^2c + c^2a)(ab+bc+ca) \leq 9$$

Solution:

Remark: This solution belongs to Micheal Rozenberg (aka Arqady) and can be found at: http://www.mathlinks.ro/viewtopic.php?p=1276520#1276520

Let $a+b+c = 3u$, $ab+ac+bc = 3v^2$, $abc = w^3$ and $u^2 = tv^2$.

Hence, $t \geq 1$ and $(a^2b+b^2c+c^2a)(ab+bc+ca) \leq 9 \Leftrightarrow$

$\Leftrightarrow 3u^5 \geq (a^2b+b^2c+c^2a)v^2 \Leftrightarrow$

$\Leftrightarrow 6u^5 - v^2\sum_{cyc}(a^2b+a^2c) \geq v^2\sum_{cyc}(a^2b-a^2c) \Leftrightarrow$

$\Leftrightarrow 6u^5 - 9uv^4 + 3v^2w^3 \geq (a-b)(a-c)(b-c)v^2$.

$(a-b)^2(a-c)^2(b-c)^2 \geq 0$ gives $w^3 \geq 3uv^2 - 2u^3 - 2\sqrt{(u^2-v^2)^3}$.

Hence, $2u^5 - 3uv^4 + v^2w^3 \geq 2u^5 - 3uv^4 + v^2\left(3uv^2 - 2u^3 - 2\sqrt{(u^2-v^2)^3}\right) \geq 0$ because

$2u^5 - 3uv^4 + v^2\left(3uv^2 - 2u^3 - 2\sqrt{(u^2-v^2)^3}\right) \geq 0 \Leftrightarrow$

$\Leftrightarrow u^5 - u^3v^2 \geq v^2\sqrt{(u^2-v^2)^3} \Leftrightarrow t^3 - t + 1 \geq 0$, which is true.

Hence, $6u^5 - 9uv^4 + 3v^2w^3 \geq 0$ and enough to prove that

$(6u^5 - 9uv^4 + 3v^2w^3)^2 \geq v^4(a-b)^2(a-c)^2(b-c)^2$.

Since, $(a-b)^2(a-c)^2(b-c)^2 = 27(3u^2v^4 - 4v^6 + 6uv^2w^3 - 4u^3w^3 - w^6)$
we obtain $(6u^5 - 9uv^4 + 3v^2w^3)^2 \geq v^4(a-b)^2(a-c)^2(b-c)^2 \Leftrightarrow$
$\Leftrightarrow v^4w^6 + uv^2(u^4 + 3u^2 - 6v^4)w^3 + u^{10} - 3u^6v^4 + 3v^{10} \geq 0$.
Id est, it remains to prove that $u^2v^4(u^4 + 3u^2 - 6v^4)^2 - 4v^4(u^{10} - 3u^6v^4 + 3v^{10}) \leq 0$.
But $u^2v^4(u^4 + 3u^2 - 6v^4)^2 - 4v^4(u^{10} - 3u^6v^4 + 3v^{10}) \leq 0 \Leftrightarrow$
$\Leftrightarrow t(t^2 + 3t - 6)^2 - 4(t^5 - 3t^3 + 3) \leq 0 \Leftrightarrow (t-1)^2(t^3 - 4t + 4) \geq 0$, which is true.
Done!

Unknown origin: If x, y, z are non-negative reals such that $xy + yz + zx = 1$, then:
$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{5}{2}$

Solution:

The constraint is $3v^2 = 1$, and after multiplying with $(x+y)(y+z)(z+x)$ we the ineq is a symmetric polynomial of degree 3. Since the constraint doesn't involve w^3, and $3 \leq 5$ we only have to prove it for $x = y$ and $x = 0$. These are left for the reader. $\square$

This one is equivalent to the following: Given the non-obtuse $\triangle ABC$ with sides a, b, c and circumradius R, prove the following:
$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{a} \geq 5R$
Do you see why?

Unknown origin: Let $a, b, c \geq 1$ and $a + b + c = 9$. Prove that
$\sqrt{ab + bc + ca} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$.

Solution: Let $a = (1+x)^2, b = (1+y)^2, c = (1+z)^2$ such that $x, y, z \geq 0$. Introduce $3u = x + y + z, 3v^2 = xy + yz + zx, w^3 = xyz$. Then the constraint is $a + b + c = (a + b + c)^2 - 2(ab + bc + ca) + 2(a + b + c) + 3 = (3u)^2 - 6v^2 + 6u + 3 = 9 \iff 3u^2 + 2u - 2v^2 = 3$. After squaring we have to prove:
$\sum_{cyc}(1+x)^2(1+y)^2 \geq \left(\sum_{cyc}(1+x)\right)^2$

Since this is a symmetric polynomial inequality of degree ≤ 5 with a constraint not involving w^3, we only have to prove it for $x = y$ and $x = 0$. That is $b = a, c = 9 - 2a, a \in [1; 4]$ and $a = 1$.

These cases are left to the reader. (Note: The constraint $a, b, c \geq 1$ are far from the best. Actually the best is about $a, b, c \geq 0.1169..$) $\square$

Unknown origin: For every $a, b, c \geq 0$, no two which are zero, the following holds:

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \geq \frac{10}{(a+b+c)^2}$$

Solution: It's equivalent to:
$(a+b+c)^2\left(\sum_{cyc}(a^2+b^2)(b^2+c^2)\right) - 10(a^2+b^2)(b^2+c^2)(c^2+a^2) \geq 0$
Note that:
$\sum_{cyc}(a^2+b^2)(b^2+c^2) = \left(\sum_{cyc} b^4 + 2a^2b^2\right) + \sum_{cyc}(ab)^2 = (a^2+b^2+c^2)^2 + 9v^4 - 6uw^3 = (9u^2 - 6v^2)^2 + 9v^4 - 6uw^3$

And $(a+b+c)^2 = 9u^2$, and:

$(a^2+b^2)(b^2+c^2)(c^2+a^2) = (a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2) - a^2b^2c^2 = (9u^2-6v^2)(9v^4-6uw^3) - w^6 = 9(9u^2v^4 - 6v^6 - 6u^3w^3 + 4uv^2w^3) - w^6$

Hence we should prove:

$27u^2(27u^4 - 36u^2v^2 + 15v^4 - 2uw^3) - 90(9u^2v^4 - 6v^6 - 6u^3w^3 + 4uv^2w^3) + 10w^6 \geq 0$

Or:

$g(w^3) = 10w^6 + w^3(486u^3 - 360uv^2) + E \geq 0$

where $E = 27u^2(27u^4 - 36u^2v^2 + 15v^4) - 90(9u^2v^4 - 6v^6)$

Then $g'(w^3) = 20w^3 + 486u^3 - 360uv^2 = 20w^3 + 126u^3 + 360u(u^2-v^2) \geq 0$

(Since $u, v^2, w^3 \geq 0$ and $u^2 \geq v^2$)

Hence the expression is increasing in w^3. If we fix u, v^2 we only have to prove it for a minimized value of w^3, but w^3 is minimized only when two of a, b, c are equal or when one of a, b, c are zero.

First the case where two of a, b, c are equal.(Assume that $a, b, c > 0$ since that will be the other case) Wlog because of symmetry assume $a = b$. Define $t = \frac{a}{c}$. Upon multiplying with c^2 we only have to prove:

$\frac{1}{2t^2} + \frac{2}{t^2+1} \geq \frac{10}{(2t+1)^2} \iff$

$f(t) = 20t^3 - 11t^2 + 4t + 1 \geq 0$

$f'(t) = 60t^2 - 22t + 4 = 60\left(t - \frac{11}{60}\right)^2 + \frac{119}{60} > 0$

Since $f(t)$ is increasing it shows that $f(t) \geq f(0) = 1 > 0 \forall t \in (0; +\infty)$

Now the case where one of a, b, c are zero. Wlog because of symmetry assume that $c = 0$. Define $t = \frac{a}{b}$. After multiplying with b^2 we have to prove:

$\frac{1}{t^2+1} + \frac{1}{t^2} + 1 \geq \frac{10}{(t+1)^2} \iff$

$t^6 + 2t^5 - 6t^4 + 6t^3 - 6t^2 + 2t + 1 = (t-1)^2(t^4 + 3t^3 + 4t^2 + 3t + 1) \geq 0$

Which is obvious :) Equality iff $a = b, c = 0$ or permutations. $\square$

Hungktn:

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2$$

Solution: Writing in terms of u, v^2, w^3 we should prove:

$\frac{3u^2-2v^2}{v^2} + \frac{8w^3}{9uv^2-w^3} \geq 2 \iff$

$\frac{u^2}{v^2} + \frac{24uv^2}{9uv^2-w^3} \geq 12$

Fixing u, v^2 it is thus clearly enough to prove it when w^3 is minimized. So we will prove it when two of a, b, c are equal and when one of them are zero. Assume wlog because of symmetry that $a = b$. Introducing $t = \frac{c}{a}$ we only have to prove:

$\frac{2+t^2}{1+2t} + \frac{4t}{t^2+2t+1} \geq 2 \iff$

$t^2(t-1)^2 \geq 0$

Now to the other case: Assume wlog because of symmetry $a = 0$. Then it becomes:

$\frac{b^2+c^2}{bc} \geq 2$

Which is just AM-GM. Equality when $a = b = c$ or $a = b, c = 0$ and permutations. $\square$

5 Practice

Unknown origin: For all $a, b, c \geq 0$ such, prove:

$$\frac{a^4+b^4+c^4}{ab+bc+ca} + \frac{3abc}{a+b+c} \geq 2\frac{(a^2+b^2+c^2)}{3}$$

Stronger version:

$$5\frac{a^4+b^4+c^4}{ab+bc+ca} + 9\frac{3abc}{a+b+c} \geq 14\frac{(a^2+b^2+c^2)}{3}$$

Romania Junior Balkan TST '09: When $a + b + c = 3$, $a, b, c \geq 0$ prove:

$$\frac{a+3}{3a+bc} + \frac{b+3}{3b+ca} + \frac{c+3}{3c+ab} \geq 3$$

Unknown origin: When $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, $a, b, c \geq 0$ prove:

$$(1-a)(1-b)(1-c) \geq 8$$

Unknown origin: When $a, b, c \geq 0$ prove:

$$a^5 + b^5 + c^5 + abc(ab + bc + ca) \geq a^2b^2(a + b) + b^2c^2(b + c) + c^2a^2(c + a)$$

Indonesia 2nd TST, 4th Test, Number 4: When $a, b, c > 0$ and $ab + bc + ca = 3$ prove that:

$$3 + \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2} \geq \frac{a+b^2c^2}{b+c} + \frac{b+c^2a^2}{c+a} + \frac{c+a^2b^2}{a+b} \geq 3$$

Unknown origin: When $a, b, c > 0$ and $a + b + c = 3$ find the minimum of:

$$(3 + 2a^2)(3 + 2b^2)(3 + 2c^2)$$