Coordinate and Trigonometry Bashing

Justin Stevens

Introduction:

Coordinate bashing and trigonometry bashing are two very nice methods to solve geometry problems if you don't know how to synthetically solve them. These methods are more applicable to AMC 10/12, AIME, Mandelbrot, and other introductory/intermediate contests. For Olympiad problems, look at the Baycentric Coordinates in the Olympiad Article forum on artofproblemsolving.com.

Coordinate Bashing

To coordinate bash, you set points to specific coordinates, and typically have one key point being (0,0) (as it is easy to use). From here, you write all given correspondences between side lengths and angles to complete your coordinate system.

For intersection lines:

Find the equation for the lines and find the intersection point by finding the intersection of the equations. From here, you can use trigonometry to find desired angles, or the distance formula to find desired side lengths.

For perpendicular lines:

Use the fact that the slopes are negative reciprocals to find the slope of the line. If you know any other facts about the perpendicular line, you can use the $y - y_0 = m(x - x_0)$ equation.

For parallel lines:

Use the fact that they have the same slope.

For right triangles:

If a right triangle has legs a, b and hypotenuse c, put the triangle on the cartesian system with points at (0,0), (0,a), (b, 0) (and $\sqrt{a^2 + b^2} = c$ or $a^2 + b^2 = c^2$).

For equilateral triangles:

Righting the triangle in the form $(\frac{-a}{2}, \frac{-\sqrt{3}*a}{6}), (\frac{a}{2}, \frac{-\sqrt{3}*a}{6}), (0, \frac{\sqrt{3}*a}{3})$ is helpful. The circumcenter turns out to be $(0, 0)$ which is the same as the incenter and orthocenter in an equilateral triangle.

For other triangles (finding circumcenter/orthocenter):

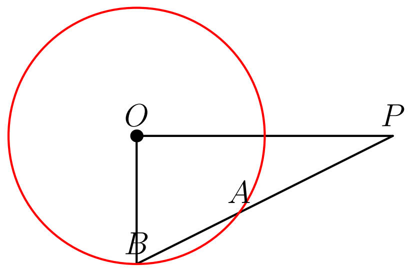
Set up equations to find the circumcenter and orthocenter. For example, if you have a triangle with coordinates (0,0), (5,0), (7,1), you could find the circumcenter by finding the perpendicular line to (0,0) and (7,1) and to (0,0) and (5,0) and finding the intersection. You can also set up the same equations with altitudes knowing the slope and a point that goes through the line.

Let's start out with an easy example and work our way up to some harder problems.

Example 1:

In circle O, $\overline{PO} \perp \overline{OB}$, and PO equals the length of the diamter of circle O. Compute $\frac{PA}{AB}$. (Source: 1999 ARML Individual Round)

Solution:



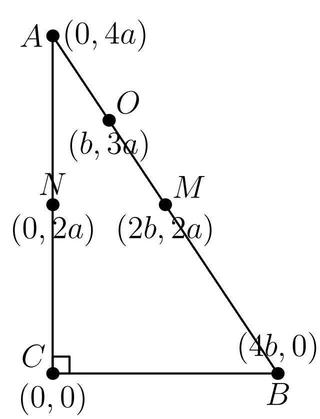
Write O as $(0,0)$, B as $(0,-r)$, and P as $(2r,0)$, where r is equal to the radius. Remark that line BP is $y = \frac{1}{2}x - r$, and we desire to find the intersection of this with $y^2 + x^2 = r^2$. Substitute this value for y into the above equation to give us $\frac{1}{4}x^2 - xr + r^2 + x^2 = r^2 \implies \frac{5}{4}x^2 = xr$ or $x = \frac{4}{5}r$ (since x can't be 0). Plug this into the equation for y to give us $y = -\frac{3}{5}r$, so the coordinates of the intersection of BP and circle O (which is A) is $(\frac{4}{5}r, \frac{-3}{5}r)$.

Now, remark that $AB = \sqrt{\frac{16}{25}r^2 + \frac{4}{25}r^2} = \frac{r}{5}\sqrt{20}$. Also, $AP = \sqrt{\frac{36}{25}r^2 + \frac{9}{25}r^2} = \frac{r}{5}\sqrt{45}$. Therefore $\frac{AP}{AB} = \frac{\frac{r}{5}\sqrt{45}}{\frac{r}{5}\sqrt{20}}$ which is the same thing as $\sqrt{\frac{45}{20}} = \boxed{\frac{3}{2}}$.

Example 2: In $\triangle ABC$, $\angle C$ is a right angle. Point M is the midpoint of $\overline{AB}$, point N is the midpoint of $\overline{AC}$, and point O is the midpoint of $\overline{AM}$ The perimeter of $\triangle ABC$ is 112 and $ON = 12.5$. What is the area of $MNCB$? (Source: Mathcounts)

Solution:

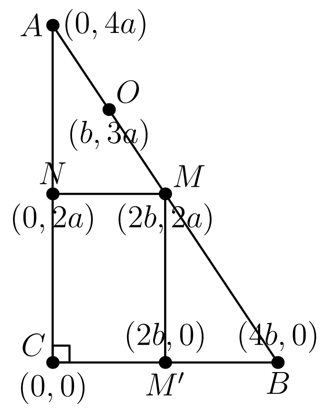
Create the following diagram:



Remark that we have $ON = 12.5$ from the problem, and using the distance formula, we get $\sqrt{a^2 + b^2} = 12.5 = \frac{25}{2}$.

Also, we have the perimeter of $\triangle ABC$ being equal to 112, so hence we must have $4a+4b+4\sqrt{a^2+b^2}=112$. This implies that $4a+4b=62 \implies a+b=\frac{31}{2}$.

From the diagram, add line NM, and draw in MM' such that $MM' \perp CB$.



Remark that $[NMCB]=[NMM'C]+[MM'B]=4ab+2ab=6ab$.

We have $a^2+b^2=(\frac{25}{2})^2$, so hence we get $(a+b)^2-2ab=(\frac{25}{2})^2 \implies (\frac{31}{2})^2-2ab=(\frac{25}{2})^2$.

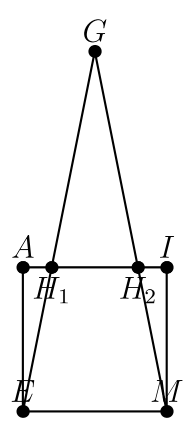
This implies that $2ab=(\frac{(31-25)(31+25)}{4}$ which implies that $2ab=(\frac{6*56}{4})=84$, so hence we have $[NMCB]=6ab=3(84)=\boxed{252}$.

Example 3:

Square $AIME$ has sides of length 10 units. Isosceles triangle GEM has base EM, and the area common to triangle GEM and square $AIME$ is 80 square units. Find the length of the altitude to EM in $\triangle GEM$. (Source: 2008 AIME I)

Solution:

Let $E=(0,0)$, $M=(10,0)$, $I=(10,10)$ and $A=(0,10)$. Let $G=(5,M)$, and we look at the following diagram:



Lemma: $\triangle AEH_1 \cong \triangle IMH_2$ Proof: $AE=IM$ from square $AIME$, $\angle EAI=\angle AIM=90°$ and

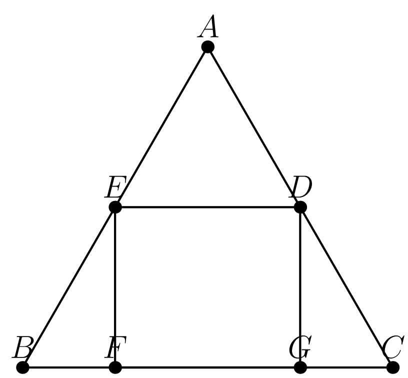
$\angle AEH_1 = 90 - \angle H_1EM = 90 - \angle H_2ME$ from isosceles $\triangle GEM$. Therefore $\triangle AEH_1 \cong \triangle IMH_2$ by AAS.

Remark that line EG is the equation $y = \frac{m}{5}x$, and hence to find the coordinate of point H_1, we use the equation $y = \frac{m}{5}x$ and $y = 10$ to give us $x = \frac{50}{m}$. Hence, the coordinate of H_1 is $(\frac{50}{m}, 10)$, and we have $[AH_1E] = \frac{1}{2} * 10 * \frac{50}{m} = [IH_2M]$. Since $[ABCD] - [AH_1E] - [IH_2M]$ is equal to the the area common to triangle GEM and square $AIME$, so hence we have $100 - 2 * \frac{1}{2} * 10 * \frac{50}{m} = 80 \implies m = 25$. We desire to find the distance from G to $\overline{EM}$, which is equal to the distance from $(5, 25)$ to $(5, 0)$ or $\boxed{25}$.

Example 4:

ABC is an equilateral triangle with side length 1. Point D lies on $\overline{AB}$, point E lies on $\overline{AC}$, and points G and F lie on $\overline{BC}$, such that $DEFG$ is a square. What is the area of $DEFG$? (Source: 2012 Stanford Math Tournament)

Solution:



Put the points on the cartesian system, where B is at point $(0, 0)$, C at $(1, 0)$ and A at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. From here, I drew the square such that we have $DG \perp BC$ and $EF \perp BC$, so that we have D above G and E above F. From here, let $G = (a, 0)$, let F be $(b, 0)$, and we are going to have D being $(a, a\sqrt{3})$ (because the equation of $\overline{AB}$ which is where D is on is $y = \sqrt{3}x$), and E being $(b, -\sqrt{3} * b + \sqrt{3})$ (because the equation of $\overline{AC}$ where E is on is $y = -\sqrt{3}x + \sqrt{3}$). From $DEFG$ being a square, we have to have $DE = EF = DG$, or $b - a = -\sqrt{3}b + \sqrt{3} = \sqrt{3}a$. From this, we have $-b + 1 = a \implies b = 1 - a$. Substitute this into $b - a = \sqrt{3}a$ to give $1 - a - a = \sqrt{3}a \implies a(2 + \sqrt{3}) = 1 \implies a = \frac{1}{2+\sqrt{3}}$. From this, we know that one side length of the square is $\frac{\sqrt{3}}{2+\sqrt{3}} = \frac{\sqrt{3}(2-\sqrt{3})}{1} = 2\sqrt{3} - 3$. We need to square this to get the area, which is $(2\sqrt{3} - 3)^2 = \boxed{21 - 12\sqrt{3}}$.

Example 5:

An equilateral triangle ABC is inscribed in a circle. Points D and E are midpoints of $\overline{AB}$ and $\overline{BC}$, respectively, and F is the point where $(DE)^{\rightarrow}$ meets the circle. Find DE/EF. (Source: ARML)

Solution:

Set the circumcenter to be O, which is at the point $(0, 0)$. WLOG, we let each side length be 1, and we put the vertices from left to right as A, C, B to give us $A = (\frac{-1}{2}, \frac{-\sqrt{3}}{6}), C = (0, \frac{\sqrt{3}}{3}), B = (\frac{1}{2}, -\frac{\sqrt{3}}{6})$ (note that this is derived from originally putting the points at $A = (0, 0), B = (1, 0)$ and $C = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, from which we get $O = (\frac{1}{2}, \frac{\sqrt{3}}{6})$ and then move this point to the origin.) Now remark that we desire to find $\frac{DE}{EF}$, and we are going to use a combination coordinate bashing, and similar triangles to solve this problem.

First off, remark that $OA = OC = OB$ by definition of the circumcenter of $\triangle ABC$. From this, we get the circumradius of the triangle is $\frac{1}{3}$, so hence the circle is defined by $x^2 + y^2 = \frac{1}{3}$. Also, remark that since D is the midpoint of $\overline{AB}$, we have $D = (0, \frac{-\sqrt{3}}{6})$, and similarly $E = (\frac{1}{4}, \frac{\sqrt{3}}{12})$.

Now, we find the slope of $\overline{DE}$, and use that the y intercept is $b = \frac{-\sqrt{3}}{6}$ to find the equation of this line. The slope is $\frac{\frac{\sqrt{3}}{12}+\frac{\sqrt{3}}{6}}{\frac{1}{4}} = \frac{\sqrt{3}}{4} * 4 = \sqrt{3}$. So the equation of this line is $y = \sqrt{3}x - \frac{\sqrt{3}}{6}$. Substitute this into the equation $y^2 + x^2 = \frac{1}{3}$ to give $3(x^2 - \frac{1}{3}x + \frac{1}{36}) + x^2 = \frac{1}{3} \implies 4x^2 - x + \frac{1}{12} = \frac{1}{3} \implies 4x^2 - x - \frac{1}{4} = 0$ which gives $16x^2 - 4x - 1 = 0$. From this, use the quadratic equation to find that $x = \frac{4 \pm \sqrt{16+64}}{32} = \frac{4 \pm 4\sqrt{5}}{32} = \frac{1 \pm \sqrt{5}}{8}$. Since we are looking for when $(DE)^{\rightarrow}$ intersects the circle, we use the greater value of x, to give us the x coordinate being $\frac{1+\sqrt{5}}{8}$ of the point of intersection.

Now is where we use the similar triangles. Look at segment $\overline{DF}$, and drop E down to DB such that they meet at H and $EH \perp DB$. Also, drop F down to line DB such that $DB \perp FG$. Remark that we must have $\triangle FDG \sim \triangle DEH$ by AA similarity, so hence we have $\frac{DE}{DH} = \frac{DF}{DG}$. Re-arrange this to give $\frac{DE}{DF} = \frac{DH}{DG}$. We know $DH = \frac{1}{4}$ (since $D = (0, -\frac{\sqrt{3}}{6})$ and $H = (\frac{1}{4}, -\frac{\sqrt{3}}{6})$ and $DG = \frac{1+\sqrt{5}}{8}$ (same reasonign), so hence we get $\frac{DE}{DF} = \frac{2}{1+\sqrt{5}}$. Take the reciprocal of both sides to give $\frac{DF}{DE} = \frac{1+\sqrt{5}}{2}$, and $DF = DE + EF$, so we get $\frac{DE+EF}{DE} = \frac{1+\sqrt{5}}{2} \implies \frac{EF}{DE} = \frac{\sqrt{5}-1}{2}$. Now, we take the reciprocal of both sides once more to give $\frac{DE}{EF} = \frac{2}{\sqrt{5}-1} = \frac{2(\sqrt{5}+1)}{4} = \boxed{\frac{1+\sqrt{5}}{2}}$.

Trigonometry Bashing

Trigonometry bashing is done in problems that you are trying to find angles or side measures and you do not have a lot of information on the problem. Trigonometry bashing and coordinate bashing can be used together, which is a very useful tool, or alone. Trigonometry bashing has a few key steps which I will show.

Law of sines/cosines

A very useful tool in many problems is to use the law of sines and cosines. In a triangle with sides A, B, C with angles a, b, c opposite to their respective sides, $\frac{\sin(\angle a)}{A} = \frac{\sin(\angle b)}{B} = \frac{\sin(\angle c)}{C}$.

For a triangle with the same conditions, $c^2 = a^2 + b^2 - 2ab\cos(\angle c)$

Area formula for triangle

The area of a triangle with the same conditions as before is $\frac{1}{2} \times ab\sin(c)$

Sum of angles is 180 degrees

This can be a useful tool if you have an isosceles triangle and you have side correspondences. Using the law of sines and the fact that $\sin(180^\circ - \theta) = \sin(\theta)$, you can find angle relationships with other angles.

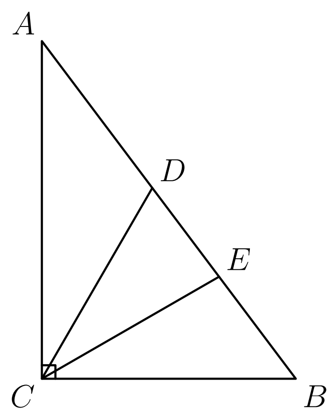
I will give two examples that show trigonometry bashing.

Example 1:

Let $\triangle ABC$ be a right triangle with right angle at C. Let D and E be points on $\overline{AB}$ with D betwen A and E such that $\overline{CD}$ and $\overline{CE}$ trisect $\angle C$. If $\frac{DE}{BE} = \frac{8}{15}$, then $\tan B$ can be written as $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers, and p is a positive integer not divisible by the square of any prime. Find $m+n+p$. (Source: 2012 AIME I)

Solution:

See the following diagram created by David Altizio:



Note that $\angle ACD = \angle DCE = \angle ECB = 30°$.

From the law of sines: $\frac{\sin(30°)}{8x} = \frac{\sin(\angle CDE)}{CE}$ and $\frac{\sin(30°)}{15x} = \frac{\sin(\angle DBC)}{CE}$

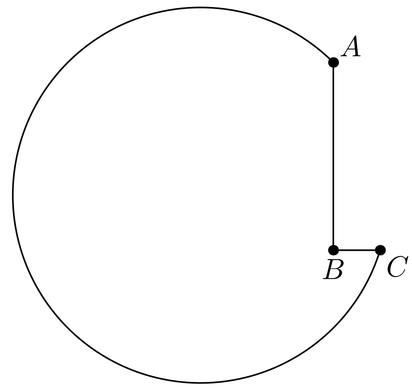
Hence $8x\sin(\angle CDE) = 15x\sin(\angle DBC) = \sin(30°)CE$ or $8\sin(\angle CDE) = 15\sin(\angle DBC)$.

Note that $\angle CDE + \angle DBC = 120°$ from $\triangle CDB$ having $\angle DCB = 60°$. Hence, we get $\angle CDE = 120° - \angle DBC$.

Substitute this in to give us $8\sin(120° - \angle DBC) = 15\sin(\angle DBC)$. This gives us
$8 * (\sin(120°)\cos(\angle DBC) - \cos(120°)\sin(\angle DBC)) = 15\sin(\angle DBC)$. This implies that
$4\sqrt{3}\cos(\angle DBC) + 4\sin(\angle DBC) = 15\sin(\angle DBC)$ which gives us:
$4\sqrt{3}\cos(\angle DBC) = 11\sin(\angle DBC) \implies \frac{\sin(\angle DBC)}{\cos(\angle DBC)} = \frac{4\sqrt{3}}{11}$ which is the same as $\tan(\angle DBC)$ or $\tan(B)$. This implies that we get an answer of $4 + 3 + 11 = \boxed{18}$.

Example 2:

A machine-shop cutting tool has the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of AB is 6 cm, and that of BC is 2 cm. The angle ABC is a right angle. Find the square of the distance (in centimeters) from B to the center of the circle.



Solution:

Let O be the center of the circle, and draw line OC. By the Pythagorean theorem, we have $AC = 2\sqrt{10}$. Note that in $\triangle AOC$, we have $\angle OAC = \angle ACO$, which we let equal to θ and $\angle AOC = \beta$. Use the law of sines to give $\frac{\sin(\theta)}{\sqrt{50}} = \frac{\sin(\beta)}{2\sqrt{10}} \implies \sin(\theta) = \frac{\sqrt{5}\sin(\beta)}{2}$. Also note that $\beta + 2\theta = \pi$ which implies $\beta = \pi - 2\theta$, so hence we get $\sin(\theta) = \frac{\sqrt{5}\sin(2\theta)}{2} \implies \sin(\theta) = \sqrt{5}\cos(\theta)\sin(\theta)$. We divide this equation by $\sin(\theta)$ since we can't have $\sin(\theta) = 0$ (this would imply $\theta = 0$), so hence we have $\cos(\theta) = \frac{1}{\sqrt{5}} = \cos(\angle OAC)$

Also, remark that $\sin(\angle BAC) = \frac{2}{2\sqrt{10}} = \frac{1}{\sqrt{10}}$ by definition.

We desire to find $\sin(\angle OAB) = \sin(\angle OAC - \angle BAC) = \sin(\angle OAC)\cos(\angle BAC) - \sin(\angle BAC)\cos(\angle OAC)$. We know that $\sin(\angle OAC) = \frac{2}{\sqrt{5}}$ and $\cos(\angle BAC) = \frac{3}{\sqrt{10}}$ (from $\sin^2(\theta) + \cos^2(\theta) = 1$). Hence, we have $\sin(\angle OAB) = \frac{2}{\sqrt{5}} * \frac{3}{\sqrt{10}} - \frac{1}{\sqrt{10}} * \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{5} * \sqrt{10}} = \frac{1}{\sqrt{2}}$. From this, we have $\angle OAB = 45°$.

Now, finally, use the Law of cosines on $\triangle ABO$ to give us $(OB)^2 = (\sqrt{50})^2 + 6^2 - 2 * \sqrt{50} * 6 * \frac{\sqrt{2}}{2} = 86 - 60 = \boxed{26}$.