Antiparallel lemma

rah4927

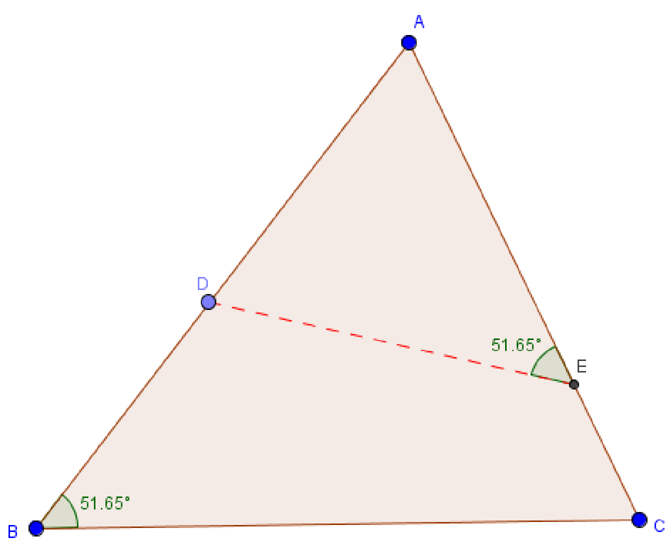
May 2016

Introduction

A huge portion of olympiad problems that we see invoke cyclic quads and their (mostly elementary) properties. In this note, we will check out an accompanying structure related to cyclic quads, namely antiparallels.

Definitions

In a $\triangle ABC$, let points D, E lie on AB and AC respectively. Then DE is an antiparallel of BC with respect to $\angle BAC$ if $\angle ADE = \angle ACB$. Note that this automatically implies $\angle AED = \angle ABC$



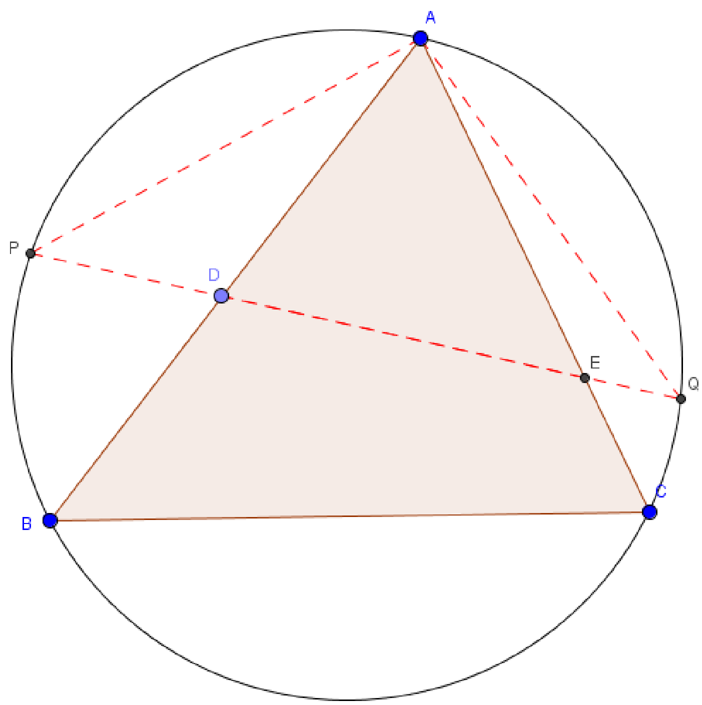
0.1 Exercise

1. (Useful) Prove that $DECB$ is cyclic.
2. Find the locus of the midpoints of antiparallels of BC .

1 The Lemma

The Statement

Let DE be an antiparallel of BC with respect to $\angle A$. Suppose DE intersects circumcircle of $\triangle ABC$ at P and Q. Then $AP = AQ$.



Proof

We show that $\triangle APB \sim ADP$. This is obvious since $\angle APB = 180 - \angle C = \angle BDE = \angle ADP$. Hence, $AP^2 = AD{\cdot}AB = AE{\cdot}AC = AQ^2$ and the conclusion follows.

Exercises

1. Rewrite the proof using directed angles.

2. Identify the other configurations of the points P and Q. What if they are on different sides of BC ?

Additional Properties

All these properties are fairly easy to prove and hence left to the reader.
1. $DECB$ is cyclic.
2. In the same triangle , two antiparallels of the same side are parallel to each other.
3. $\triangle ADE$ is similar to $\triangle ACB$.
4. (Bonus) Inversion with respect to R takes ST to its antiparallel (with respect to $\angle SRT$), wherer R, S, T are distinct points on the plane.

Problems

1. (IMO 2009) Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

2. (IMO 2015) Triangle ABC has circumcircle Ω and circumcenter O. A circle Γ with center A intersects the segment BC at points D and E, such that B, D, E, and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C, and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.
Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

3. Let the angle bisector of $\angle A$ in $\triangle ABC$ intersect the circumcircle at M . X and Y are chosen on arcs AB and AC not containing M, respectively. Let MX and MY intersect BC at P and Q. Suppose YP and XQ intersect the circumcircle again at D and E. Prove that DE is parallel to BC .