Herstein: Topics in Algebra - Homomorphisms

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21. Let S_1 and S_2 be two sets. Suppose there exists a one-to-one mapping ψ of S_1 into S_2. Show there exists an isomorphism of $A(S_1)$ into $A(S_2)$, where $A(S)$ means the set of all one-to-one mappings of S onto itself.

Proof: The mapping ψ sends a "copy" of S_1 into S_2, so if we can "extend" $A(\psi(S_1))$ in $A(S_2)$ without changing any algebraic structure it will the isomorphism or "copy" of $A(S_1)$ we need in $A(S_2)$. If $\sigma \in A(S_1)$ let's define $\phi \in A(S_2)$ in the following natural way so that ϕ is "induced" by σ:

$$\phi(s_2) = \begin{cases} s_2 \text{ if } s_2 \notin \psi(S_1). \\ \psi\sigma(s_1) \text{ if } s_2 = \psi(s_1) \text{ for some } s_1 \in S_1. \end{cases}$$

That is, ϕ moves only elements of S_2 in the image of ψ which is "our copy of S_1 in S_2". Hence, $\sigma \in A(S_1)$ induces ϕ. Let's show $\phi \in A(S_2)$ or that ϕ is one-to-one mapping of S_2 onto itself.

1. ϕ is onto:
   If $s_2 \notin \psi(S_1)$ then $\phi(s_2) = s_2$ otherwise $s_2 \in \psi(S_1)$ so $s_2 = \psi(s_1^*)$ where $s_1^* \in S_1$. Since $\sigma \in A(S_1)$ there exists $s_1^{**} \in S_1$ such that $\sigma(s_1^{**}) = s_1^*$. Let $\overline{s_2} = \psi(s_1^{**}) \in S_2$ then $\phi(\overline{s_2}) = \psi(\sigma(s_1^{**})) = \psi(s_1^*) = s_2$.

2. ϕ is one-to-one:
   Suppose $\phi(s_2) = \phi(s_2')$ and $s_2 \notin \psi(S_1)$ then $\phi(s_2) = s_2$. If $s_2' \in \psi(S_1)$ then for some $s_1'' \in S_1$, $s_2' = \psi(s_1'')$ hence $s_2 = \phi(s_2) = \phi(s_2') = \psi(\sigma(s_1''))$ which contradicts that $s_2 \notin \psi(S_1)$. Therefore, $s_2' \notin \psi(S_1)$ so that $\phi(s_2') = s_2' = \phi(s_2) = s_2$.
   Finally, we have to consider the case when $\phi(s_2) = \phi(s_2')$ and $s_2, s_2' \in \psi(S_1)$. In this case, there are $s_1', s_1'' \in S_1$ such that $\psi(s_1') = s_2$ and $\psi(s_1'') = s_2'$. Hence $\phi(s_2) = \psi(\sigma(s_1')) = \phi(s_2') = \psi(\sigma(s_1''))$. We know both ψ and σ are both one-to-one so $\psi\sigma$ is one-to-one. Therefore, $s_1' = s_1''$ and hence $s_2 = s_2'$.

Therefore, $\phi \in A(S_2)$.

Define $\eta : A(S_1) \to A(S_2)$ by $\eta(\sigma) = \phi$ as constructed above. η is the isomorphism we need. First we must show that it is a homomorphism and that it is one-to-one:

(1) η is a homomorphism:
We must show $\eta(\sigma_1\sigma_2) = \eta(\sigma_1)\eta(\sigma_2)$.
If $s_2 \notin \psi(S_1)$ then $\eta(\sigma_1\sigma_2)s_2 = \eta(\sigma_1)\eta(\sigma_2)s_2 = s_2$.

If $s_2 \in \psi(S_1)$ then $s_2 = \psi(\overline{s_1})$ for some $\overline{s_1} \in S_1$. Hence,

$\eta(\sigma_1\sigma_2)s_2 = (\psi\sigma_1\sigma_2)(\overline{s_1})$

and

$\eta(\sigma_1)\eta(\sigma_2)s_2 = \eta(\sigma_1)\psi(\sigma_2(\overline{s_1})) = \psi\sigma_1(\sigma_2(\overline{s_1})) = (\psi\sigma_1\sigma_2)(\overline{s_1})$.

Thus,

$\eta(\sigma_1\sigma_2) = \eta(\sigma_1)\eta(\sigma_2)$.

(2) η is one-to-one:
Suppose $\eta(\sigma_1) = \eta(\sigma_2)$.
Let $s_1 \in S_1$ then $\psi(s_1) \in S_2$ so if $\eta(\sigma_1) = \eta(\sigma_2)$ we have

$$\eta(\sigma_1)\psi(s_1) = \eta(\sigma_2)\psi(s_1) \implies \psi(\sigma_1(s_1)) = \psi(\sigma_2(s_1))$$

but ψ is one-to-one so $\sigma_1(s_1) = \sigma_2(s_1)$ implies $\sigma_1 = \sigma_2$, proving η is one-to-one.

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